SyDe312 (Winter 2005)

Unit 3 - Solutions

Problem 7.1 - 1

First, we plot the data points that we have. From the plot, it is reasonable to expect $f(x)$ to be a linear polynomial $f(x) = mx + b$

$$\sum_{i=1}^{11} x_i = 16.5 \quad \sum_{i=1}^{11} x_i^2 = 34.65 \quad \sum_{i=1}^{11} y_i = 22.412 \quad \sum_{i=1}^{11} x_i y_i = 54.2094$$

Then, solving for b and m using the following equations:

$$\begin{aligned} nb + (\sum_{i=1}^{n} x_i)m &= \sum_{i=1}^{n} y_i \\ (\sum_{i=1}^{n} x_i)b + (\sum_{i=1}^{n} x_i^2)m &= \sum_{i=1}^{n} x_i y_i \end{aligned}$$

As a result, $f(x) = 2.08109x - 1.083727$

Using the following equation:

$$E = \sqrt{\frac{1}{n}\sum_{i=1}^{n}[f(x_i) - y_i]^2}$$

The root-mean-square-error $E = 0.236$

The graph of $f(x)$ and original data points is shown.

Problem 7.1 - 3

The standard form for the quadratic least square is $f(x) = a_1 + a_2 x + a_3 x^2$
Here n=21:
$\sum_{j=1}^{n} x_j = 0$
$\sum_{j=1}^{n} x_j^2 = 7.7$
$\sum_{j=1}^{n} y_j = 37.733$
$\sum_{j=1}^{n} x_j y_j = -16.9269$
$\sum_{j=1}^{n} x_j^3 = 0$

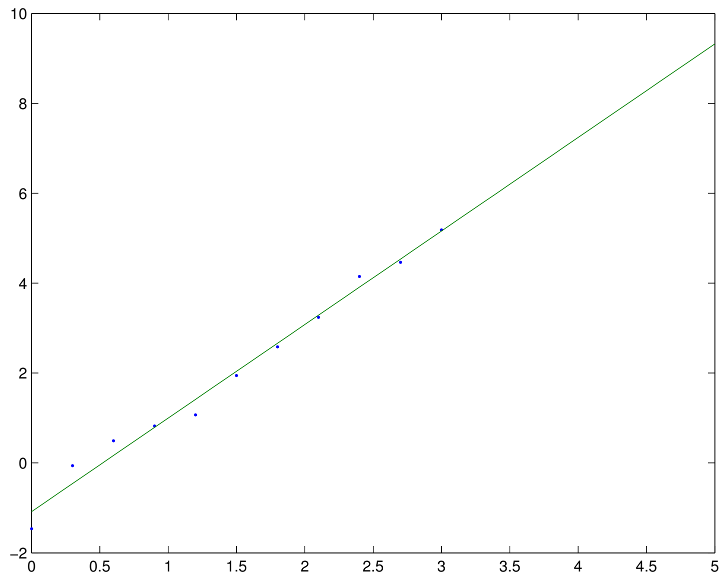

Figure 1: Least Square Fit

$\sum_{j=1}^{n} x_j^4 = 5.0666$
$\sum_{j=1}^{n} x_j^2 y_j = 29.5825$
Then, solving for a_1, a_2, and a_3 using the following equations:
$na_1 + [\sum_{j=1}^{n} x_j]a_2 + [\sum_{j=1}^{n} x_j^2]a_3 = \sum_{j=1}^{n} y_j$
$[\sum_{j=1}^{n} x_j]a_1 + [\sum_{j=1}^{n} x_j^2]a_2 + [\sum_{j=1}^{n} x_j^3]a_3 = \sum_{j=1}^{n} y_j x_j$
$[\sum_{j=1}^{n} x_j^2]a_1 + [\sum_{j=1}^{n} x_j^3]a_2 + [\sum_{j=1}^{n} x_j^4]a_3 = \sum_{j=1}^{n} y_j x_j^2$

As a result, $f(x) = 7.3301079x^2 - 2.2158961x - 0.98132527$
Using the following equation:

$E = \sqrt{\frac{1}{n}\sum_{i=1}^{n}[f(x_i) - y_i]^2}$
The root-mean-square-error $E = 0.345$
The graph of $f(x)$ and the data points is given below

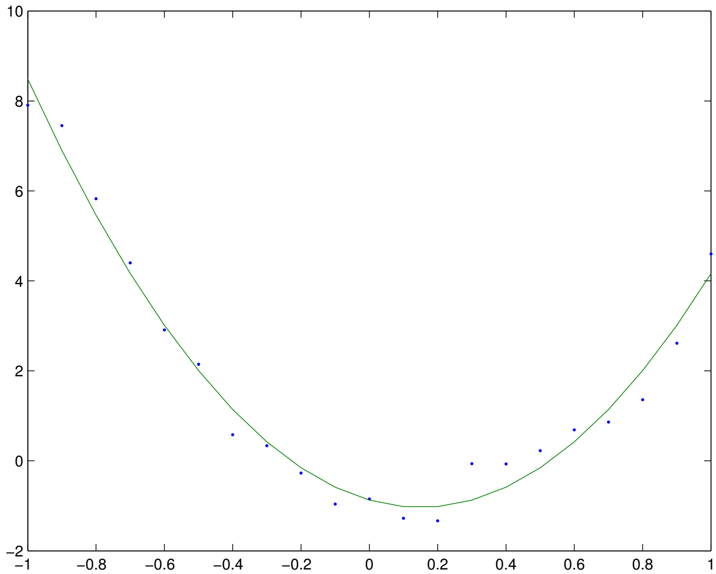

Figure 2: Quadratic least squares polynomial fit.

Problem 4.1 - 2

a)Substituting the values in the function:

$$\begin{aligned} P(0) &= a+b=2 \\ P(0.5) &= a+c=5 \\ P(1) &= a-b=4 \end{aligned}$$

Solving those equations, we get $a=3, b=-1, c=2$. The function is then

$$P(x)=3-\cos\pi x+2\sin\pi x$$

b) Using the Lagrange quadratic interpolating polynomial:
$P_2(x)=y_0L_0(x)+y_1L_1(x)+y_2L_2(x)$ where
$L_0(x)=\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}$
$L_1(x)=\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}$
$L_2(x)=\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$
$P_2(x)=\frac{(x-0.5)(x-1)}{-0.5}(2)+\frac{(x)(x-1)}{-0.25}(5)+\frac{(x)(x-0.5)}{0.5}(4)$
$P_2(x)=-8x^2+10x+2$

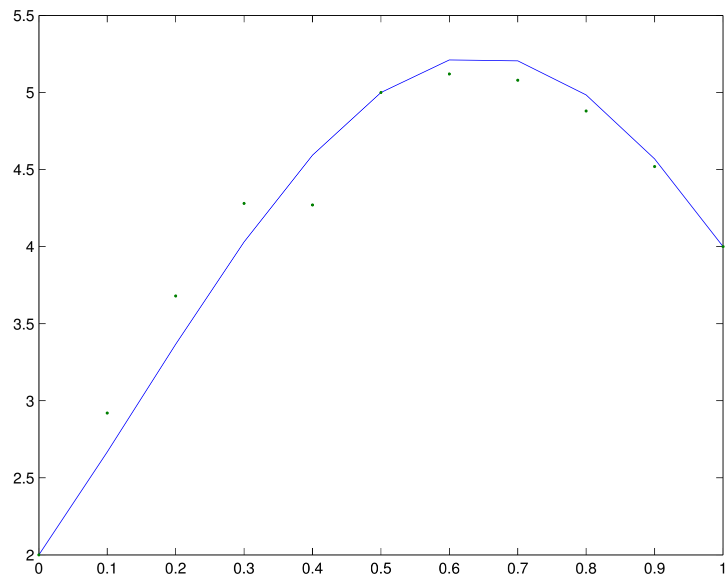

Figure 3: Quadratic Polymomial Interpolation

Problem 4.1 - 3

Each two adjacent points are to be taken together, using the following equation:

$$P_1(x) = \frac{(x_1 - x)y_0 + (x - x_0)y_1}{(x_1 - x_0)}$$

For example, using $(0.0, 2.0000)$ and $(1.0, 2.1592)$ gives:

$$\begin{aligned} P_1(x) &= \frac{(1.0 - x)2.0000 + (x - 0.0)2.1592}{(1.0 - 0.0)} \\ &= 2 + 0.1592x \end{aligned}$$

Any x value between 0 and 1 should satisfy this equation. Likewise, we do for the remaining points. The continuous graph of $f(x)$ is shown.

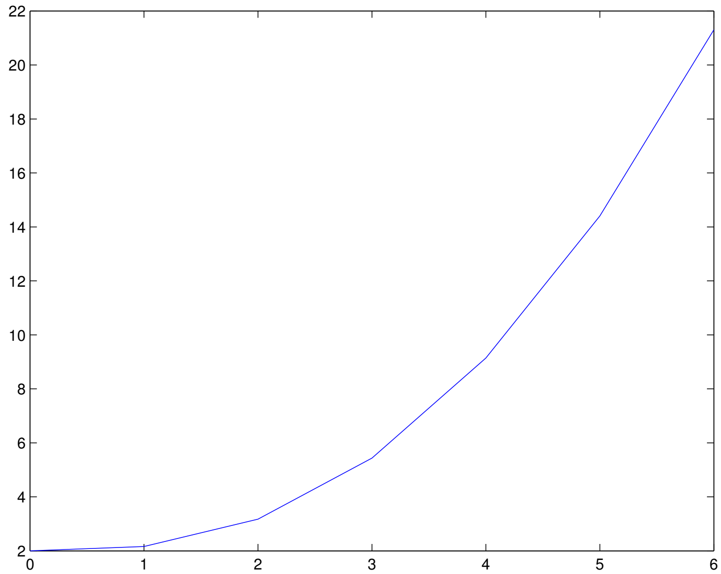

Figure 4: Linear Interpolation

Problem 4.1 - 6

We have:

$$
\begin{aligned}
P_2(x) &= \frac{(x-1)(x-2)}{2}(-1) + \frac{x(x-2)}{-1}(-1) + \frac{x(x-1)}{2}(7) \\
&= \frac{-x^2+3x-2}{2} + \frac{2x^2-4x}{2} + \frac{7x^2-7x}{2} \\
&= \frac{8x^2-8x-2}{2} = 4x^2-4x-1
\end{aligned}
$$

Problem 4.1 - 7

Using the same formulas from problem 2:

$$
\begin{aligned}
P_2(x) &= \frac{x(x+1)}{2}(-15) + \frac{x(x+2)}{-1}(-8) + \frac{(x+1)(x+2)}{2}(-3) \\
&= -x^2+4x-3
\end{aligned}
$$

To find the zeros, let $P_2(x) = -x^2+4x-3 = 0$. This gives $(-x+3)(x-1) = 0$ so $x = 3$ or $x = 1$.

We know that it is zero at 1 and 3, we test it for any point in between. Then we conclude that it does have a max and it is at $P_2(2) = 1$

Problem 4.1 - 12

a) Here $n = 3$

$$P_3(x) = L_0(x) + L_1(x) + L_2(x) + L_3(3)$$

It is a polynomial of degree at most 3. The interpolation is $f(x) = 1$. $f(x)$ is a polynomial of degree 0, so we must have $f(x) = P_0(x) = P_1(x) = P_2(x) = P_3(x) = 1$

b)The proof to any arbitrary degree $n > 0$ is similar to the one in part (a).

Problem 4.1 - 24

$f[x_0, x_1] = \frac{f(x_1)-f(x_0)}{x_1-x_0}$
$f[x_1, x_2] = \frac{f(x_2)-f(x_1)}{x_2-x_1}$
$f[x_0, x_1, x_2] = \frac{f[x_1,x_2]-f[x_0-x_1]}{x_2-x_0}$
This gives:
$f[x_0, x_1] = \frac{0.24-0.20)}{0.2-0.1} = 0.4$
$f[x_1, x_2] = \frac{0.30-0.24)}{0.3-0.2} = 0.6$
$f[x_0, x_1, x_2] = \frac{0.6-0.4}{0.3-0.1} = 1.0$
$P_1(x) = f(x_0) + (x - x_0)f[x_0, x_1]$
$P_2(x) = f(x_0) + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2]$
This gives:
$P_1(0.15) = 0.20 + (0.15 - 0.1)(0.4) = 0.22$
$P_2(0.15) = 0.20 + (0.15 - 0.1)(0.4) + (0.15 - 0.1)(0.15 - 0.20)(1.0) = 0.2175$

Problem 4.1 - 25

$f(x) = \frac{1}{1+x}$
$x_0 = 0, x_1 = 1, x_2 = 2$
Then, $f(0) = 1, f(1) = \frac{1}{2}, f(2) = \frac{1}{3}$
$f[x_0, x_1] = \frac{f(x_1)-f(x_0)}{x_1-x_0}$
$f[x_1, x_2] = \frac{f(x_2)-f(x_1)}{x_2-x_1}$

$f[x_0, x_1, x_2] = \frac{f[x_1,x_2]-f[x_0-x_1]}{x_2-x_0}$
This gives:

$$\begin{aligned} f[x_0, x_1] &= \frac{0.5-1.0}{1.0} = -\frac{1}{2} \\ f[x_1, x_2] &= \frac{\frac{1}{3}-\frac{1}{2}}{2.0-1.0} = -\frac{1}{6} \\ f[x_0, x_1, x_2] &= \frac{-\frac{1}{6}+\frac{1}{2}}{2.0-0} = \frac{1}{6} \end{aligned}$$

The quadratic polynomial that interpolates $f(x)$ is

$$\begin{aligned} P_2(x) &= f(x_0) + (x-x_0)f[x_0, x_1] + (x-x_0)(x-x_1)f[x_0, x_1, x_2] \\ &= 1 - \frac{1}{2} + \frac{1}{6}x(x-1) = \frac{1}{6}x^2 - \frac{2}{3}x + 1 \end{aligned}$$

On the interval $[0, 2]$, the graph of $f(x) - P_2(x)$ is shown below.

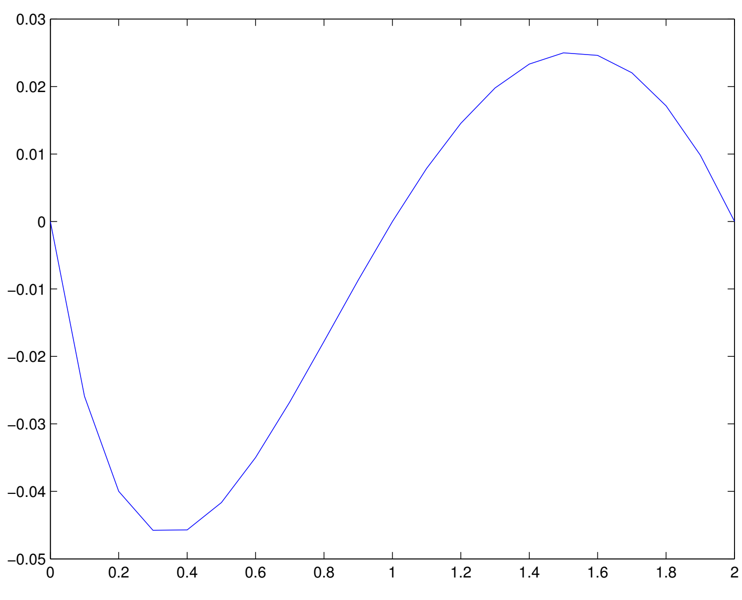

Figure 5: Error of $f(x) - P_2(x)$

Problem 4.1 - 28

Calculating the divided differences as we did in problems 24 and 25, we got the following table:

x_i	-2	-1	0	1	2	3
$p(x_i)$	-5	1	1	1	7	25
$p[x_i, x_{1+1}]$	6	0	0	6	18	
$p[x_i, \ldots, x_{i+2}]$	-3	0	3	6		
$p[x_i, \ldots, x_{i+3}]$	1	1	1			
$p[x_i, \ldots, x_{i+4}]$	0	0				

So the polynomial is:
$p(x) = -5 + 6(x+2) - 3(x+2)(x+1) + (x+2)(x+1)x = x^3 - x + 1$
The degree of the polynomial is 3.

Problem 4.3 - 1

a) We have $(0,1), (1,1), (2,5)$
We plot them and connect them with straight lines
Between $(0,1)$ and $(1,1)$, $l(x) = 1$.
Between $(1,1)$ and $(2,5)$, we need to find the equation
Slope$=\frac{5-1}{2-1} = 4$
Straight line equation is $y = mx + b$
This implies $1 = (4)(1) + b$
As a result $b = -3$
Then:
$$l(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 4x - 3 & 1 \leq x \leq 2 \end{cases}$$

b) As done in problem 2b of section 4.1:
Using the formula for the Lagrange quadratic interpolating polynomial:
$P_2(x) = y_0 L_0(x) + y_1 L_1(x) + y_2 L_2(x)$ where
$L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}$
$L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}$
$L_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$
$P_2(x) = \frac{x^2-3x+2}{2}(1) + \frac{x^2-2x}{-1}(1) + \frac{x^2-x}{2}(5)$
$P_2(x) = -2x^2 - 2x + 1$

c)Use the following equation:

$$\frac{x_j - x_{j-1}}{6}M_{j-1} + \frac{x_{j+1} - x_{j-1}}{3}M_j + \frac{x_{j+1} - x_j}{6}M_{j+1} = \frac{y_{j+1} - y_j}{x_{j+1} - x_j} - \frac{y_j - y_{j-1}}{x_j - x_{j-1}}$$

This will give $\frac{1}{6}M_1 + \frac{2}{3}M_2 + \frac{1}{6}M_3 = 4$ with solution $M_1 = M_3 = 0, M_2 = 6$.

Substituting these values in this equation:

$$s(x) = \frac{(x_j - x)^3 M_{j-1} + (x - x_{j-1})^3 M_j}{6(x_j - x_{j-1})} + \frac{(x_j - x)y_{j-1} + (x - x_{j-1})y_j}{x_j - x_{j-1}} - \frac{1}{6}(x_j - x_{j-1})[(x_j - x)M_{j-1} + (x - x_{j-1})M_j$$

Finally, we get:

$$s(x) = \begin{cases} x^3 - x + 1 & 0 \le x \le 1 \\ -x^3 + 6x^2 - 7x + 3 & 1 \le x \le 2 \end{cases}$$

The graph is shown below

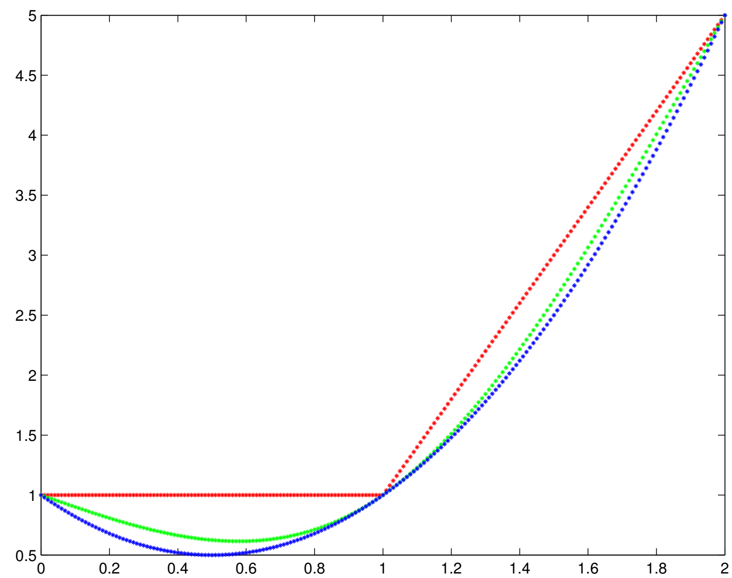

Figure 6: Interpolating functions: red:$l(x)$, green:$s(x)$, blue:$P_2(x)$

Problem 4.3 - 3

a) We have (0,0), (0.5,0.25), (1,1), (2,-1), (3,-1)
We plot them and connect them with straight lines

As done in problem 1, we need to find the equation for each segment
As a result we get:

$$l(x) = \begin{cases} \frac{x}{2} & 0 \le x \le \frac{1}{2} \\ \frac{3x-1}{2} & \frac{1}{2} \le x \le 1 \\ -2x+3 & 1 \le x \le 2 \\ -1 & 2 \le x \le 3 \end{cases}$$

b) As done in problem 2 part b of section 4.1:
dividing the interval on sets of adjacent points
then using the Lagrange quadratic interpolating polynomial:
$P_2(x) = y_0 L_0(x) + y_1 L_1(x) + y_2 L_2(x)$ where
$L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}$
$L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}$
$L_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$

First for the interval [0,1]

$$P_2(x) = (x^2 - 3x + 2)(0) + (-4x^2 + 4x)(\frac{1}{4}) + (2x^2 - x)(1) = x^2$$

and for the interval [1,3]

$$P_2(x) = \frac{x^2 - 5x + 6}{2}(1) + \frac{x^2 - 4x + 3}{-1}(-1) + \frac{x^2 - 3x + 2}{2}(-1) = x^2 - 5x + 5$$

Then we have:

$$q(x) = \begin{cases} x^2 & 0 \le x \le 1 \\ x^2 - 5x + 5 & 1 \le x \le 3 \end{cases}$$

c) Using the following equation:

$$\frac{x_j - x_{j-1}}{6} M_{j-1} + \frac{x_{j+1} - x_{j-1}}{3} M_j + \frac{x_{j+1} - x_j}{6} M_{j+1} = \frac{y_{j+1} - y_j}{x_{j+1} - x_j} - \frac{y_j - y_{j-1}}{x_j - x_{j-1}}$$

gives:

$$\begin{aligned} M_1 + 4M_2 + M_3 &= 12 \\ M_2 + 6M_3 + 2M_4 &= -42 \\ M_3 + 4M_4 + M_5 &= 12 \end{aligned}$$

with solution: $M_1 = M_5 = 0$ and $M_2 = M_4 = 38/7, M_3 = -68/7$.

Substituting these values in this equation:

$$s(x) = \frac{(x_j - x)^3 M_{j-1} + (x - x_{j-1})^3 M_j}{6(x_j - x_{j-1})} + \frac{(x_j - x)y_{j-1} + (x - x_{j-1})y_j}{x_j - x_{j-1}} - \frac{1}{6}(x_j - x_{j-1})[(x_j - x)M_{j-1} + (x - x_{j-1})M_j$$

gives:

$$s(x) = \begin{cases} \frac{38}{21}x^3 + \frac{1}{21}x & 0 \le x \le \frac{1}{2} \\ -\frac{106}{21}x^3 + \frac{72}{7}x^2 - \frac{107}{21}x + \frac{6}{7} & \frac{1}{2} \le x \le 1 \\ \frac{53}{21}x^3 - \frac{87}{7}x^2 + \frac{370}{21}x - \frac{47}{7} & 1 \le x \le 2 \\ -\frac{19}{21}x^3 + \frac{57}{7}x^2 - \frac{494}{21}x + \frac{145}{7} & 2 \le x \le 3 \end{cases}$$

d) Using the following equation:

$$\frac{x_j - x_{j-1}}{6}M_{j-1} + \frac{x_{j+1} - x_{j-1}}{3}M_j + \frac{x_{j+1} - x_j}{6}M_{j+1} = \frac{y_{j+1} - y_j}{x_{j+1} - x_j} - \frac{y_j - y_{j-1}}{x_j - x_{j-1}}$$

gives $\frac{1}{6}M_1 + M_2 + \frac{1}{3}M_3 = -2$
It is given that:
$x_1 = 0$, $x_2 = 1$, $x_3 = 3$, $z_1 = \frac{1}{2}$, $z_2 = 2$
Using the relation $s(z_1) = f(z_1)$, $s(z_2) = f(z_2)$ gives

$$\begin{aligned} M_1 + M_2 &= 4 \\ M_2 + M_3 &= 4 \end{aligned}$$

with solution $M_1 = M_3 = 12$, $M_2 = -8$. Substituting in this formula:

$$s(x) = \frac{(x_j - x)^3 M_{j-1} + (x - x_{j-1})^3 M_j}{6(x_j - x_{j-1})} + \frac{(x_j - x)y_{j-1} + (x - x_{j-1})y_j}{x_j - x_{j-1}} - \frac{1}{6}(x_j - x_{j-1})[(x_j - x)M_{j-1} + (x - x_{j-1})M_j$$

gives $s(x) = \begin{cases} \frac{-10x^3 + 18x^2 - 5x}{3} & 0 \le x \le 1 \\ \frac{5x^3 - 27x^2 + 40x - 15}{3} & 1 \le x \le 3 \end{cases}$

The graph is shown below

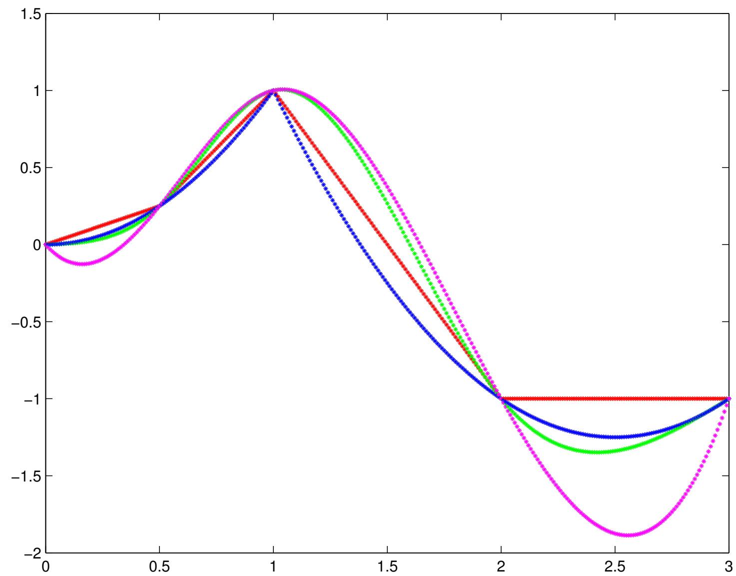

Figure 7: Interpolating functions: red:$l(x)$, green:$s(x)$, blue:$q(x)$, pink: not-a-knot

Problem 4.3 - 7

a) We have (1,1), (2,$\frac{1}{2}$), (3,$\frac{1}{3}$), (4,$\frac{1}{4}$)
$n = 4$ and all $x_j - x_{j-1} = 1$
Using this equation:

$$\frac{x_j - x_{j-1}}{6} M_{j-1} + \frac{x_{j+1} - x_{j-1}}{3} M_j + \frac{x_{j+1} - x_j}{6} M_{j+1} = \frac{y_{j+1} - y_j}{x_{j+1} - x_j} - \frac{y_j - y_{j-1}}{x_j - x_{j-1}}$$

gives

$$\begin{aligned} \frac{1}{6} M_1 + \frac{2}{3} M_2 + \frac{1}{6} M_3 &= \frac{1}{3} \\ \frac{1}{6} M_2 + \frac{2}{3} M_3 + \frac{1}{6} M_4 &= \frac{1}{12} \end{aligned}$$

Differentiating the following equation:

$$s(x) = \frac{(x_j - x)^3 M_{j-1} + (x - x_{j-1})^3 M_j}{6(x_j - x_{j-1})} + \frac{(x_j - x) y_{j-1} + (x - x_{j-1}) y_j}{x_j - x_{j-1}} - \frac{1}{6}(x_j - x_{j-1})[(x_j - x) M_{j-1} + (x - x_{j-1}) M_j$$

We get:

$$s'(x) = \frac{-3(x_j - x)^2 M_{j-1} + 3(x - x_{j-1})^2 M_j}{6(x_j - x_{j-1})} + \frac{y_j - y_{j-1}}{x_j - x_{j-1}} - \frac{1}{6}(x_j - x_{j-1})(M_j - M_{j-1})$$

The condition $s'(x_1) = f'(x_1)$ with $j = 2$ gives

$$\frac{-3(x_2 - x_1)^2 M_1}{6(x_2 - x_1)} + \frac{y_2 - y_1}{x_2 - x_1} - \frac{1}{6}(x_2 - x_1)(M_2 - M_1) = f'(x_1)$$

Rearranging and multiplying both sides by -1 gives:

$$\frac{x_2 - x_1}{3} M_1 + \frac{x_2 - x_1}{6} M_2 = \frac{y_2 - y_1}{x_2 - x_1} - f'(x_1)$$

Then repeating the same thing but with $j = n$
This gives:

$$\frac{x_n - x_{n-1}}{6} M_{n-1} + \frac{x_n - x_{n-1}}{3} M_n = f'(x_n) - \frac{y_n - y_{n-1}}{x_n - x_{n-1}}$$

Substituting the values in those two functions and combining them with the first two after multyping them with 6, we get:

$$\begin{aligned} 2M_1 + M_2 &= 3 \\ M_3 + 2M_4 &= \frac{1}{8} \\ M_1 + 4M_2 + M_3 &= 2 \\ M_2 + 4M_3 + M_4 &= \frac{1}{2} \end{aligned}$$

Solving those equations, we get: $M_1 = 173/120$, $M_2 = 7/60$, $M_3 = 11/120$, $M_4 = 1/60$

Again, using the following equation:

$$s(x) = \frac{(x_j - x)^3 M_{j-1} + (x - x_{j-1})^3 M_j}{6(x_j - x_{j-1})} + \frac{(x_j - x) y_{j-1} + (x - x_{j-1}) y_j}{x_j - x_{j-1}} - \frac{1}{6}(x_j - x_{j-1})[(x_j - x) M_{j-1} + (x - x_{j-1}) M_j$$

we get:

$$s(x) = \begin{cases} \frac{-53x^3 + 332x^2 - 745x + 706}{240} & 1 \le x \le 2 \\ \frac{-x^3 + 20x^2 - 121x + 290}{240} & 2 \le x \le 3 \\ \frac{-3x^3 + 38x^2 - 175x + 344}{240} & 3 \le x \le 4 \end{cases}$$

b) The calculated samples for both natural and cubic spline are shown in the following table

x_i	Natural spline $s_n(x)$	Error $\frac{1}{x} - s_n(x)$	(a) spline $s_n(x)$	Error$\frac{1}{x} - s_n(x)$
1.25	0.85547	-5.55E-2	0.79160	8.40E-3
1.50	0.71875	-5.21E-2	0.65260	1.41E-2
1.75	0.59766	-2.62E-2	0.56230	9.12E-3
2.25	0.43099	1.35E-2	0.44837	-3.93E-3
2.50	0.38542	1.46E-2	0.40365	-3.65E-3
2.75	0.35547	8.17E-3	0.36543	-1.79E-3
3.25	0.31250	-4.81E-3	0.30684	8.56E-4
3.50	0.29167	-5.95E-3	0.28490	8.18E-4
3.75	0.27083	-4.17E-3	0.26634	3.26E-4

We could conclude that the cubic spline with the addition of the endpoint derivative conditions performs better than the natural spline. This is because $f''(x) = 2/x^3 \neq 0$.

Problem 4.3 - 14

We take the first derivative of $s(x)$

$$s'(x) = \begin{cases} 3x^2 & 0 \leq x \leq 1 \\ 2 & 1 \leq x \leq 2 \\ 6x & 2 \leq x \leq 3 \end{cases}$$

We simply need to test if $s'(x)$ is continuous. For example, $s'(0.5) = 0.75$ while $s'(1.5) = 2$. Since $s'(0.5) \neq s'(1.5)$, $s'(x)$ is not continous on $x = 1$. Thus, $s(x)$ is not a cubic spline on the interval $[0, 3]$.

SyDe 312 Unit III: Extra curve fitting problems

In the suggested text problems, and the problems below, use a variety of methods (normal equations+Cholesky, QR factorization, SVD) to find solutions to the least-squares problem. Make sure you can confidently apply any of those methods. Plot the data and fitting curves in Matlab to examine the relationship between them.

1. Find the power fits $y = a/x$ and $y = b/x^2$ for the following data, and check residuals to determine which fit is best:
   $x = [\ 0.5 \quad 0.8 \quad 1.1 \quad 1.8 \quad 4.0\ ]$
   $y = [\ 7.1 \quad 4.4 \quad 3.2 \quad 1.9 \quad 0.9\ ]$.

2. Find the least-squares parabola $y = ax^2 + bx + c$ and for the following data:
   $x = [\ -2 \quad -1 \quad 0 \quad 1 \quad 2\ ]$
   $y = [\ 2.8 \quad 2.1 \quad 3.3 \quad 6.0 \quad 11.5\ ]$.

3. For the given data, find the following least-squares curves: (a) $y = ce^{ax}$; and (b) $y = cx^a$. Check residuals to determine which fit is better:
   $x = [\ 1 \quad 2 \quad 3 \quad 4 \quad 5\ ]$
   $y = [\ 0.6 \quad 1.9 \quad 4.3 \quad 7.6 \quad 12.6\ ]$.

4. For the following data find an appropriate fitting function (linear, quadratic, cubic, exponential, linear reciprocal, quadratic reciprocal):

   (a) $x = [\ 3.0 \quad 0.5 \quad 6.9 \quad 6.5 \quad 9.8 \quad 5.5 \quad 4.0 \quad 2.0 \quad 6.3 \quad 7.3\ ]$
   $y = [\ 5.8 \quad -2.5 \quad 22.1 \quad 23.9 \quad 36.5 \quad 18.0 \quad 10.3 \quad 2.2 \quad 18.4 \quad 21.5\ ]$.

   (b) $x = [\ 8.80 \quad 4.90 \quad 8.90 \quad 7.60 \quad 6.60 \quad 9.70 \quad 1.70 \quad 1.40 \quad 7.60 \quad 3.10\ ]$
   $y = [\ -0.02 \quad -0.04 \quad -0.02 \quad -0.02 \quad -0.03 \quad -0.02 \quad -0.11 \quad -0.13 \quad -0.03 \quad -0.06\ ]$.

   (c) $x = [\ 4.80 \quad 4.00 \quad -4.70 \quad -2.20 \quad -3.10 \quad 1.10 \quad -2.90 \quad 5.00 \quad -4.80 \quad -0.60\ ]$
   $y = [\ 262.00 \quad 156.10 \quad -200.60 \quad -27.00 \quad -75.20 \quad 9.10 \quad -62.40 \quad 331.30 \quad -262.70 \quad -2.60\ ]$.

Additional Problems

March 1, 2004

1 Problem 1

To find the coefficient in the equation y = a/x

$$e = \sum_1^n (y_i - a/x_i)^2$$
$$de/da = \sum_1^n 2(y_i - a/x_i)(-1/x_i) = 0$$
$$\sum_1^n (1/x_i)(y_i - a/x_i) = 0$$
$$a = \sum_i^n (x_i y_i)/n = 3.552$$
$$res = 0.0067$$

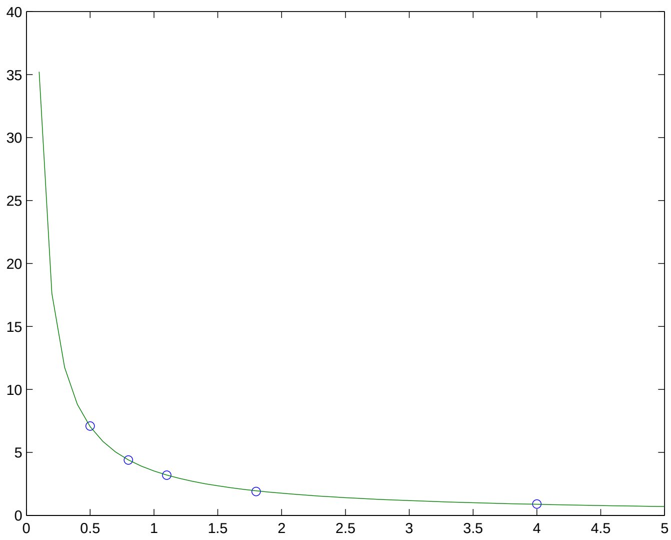

Figure 1: y = a/x

Following the same steps for the equation $y = b/x^2$

$$b = \sum_i^n (x_i^2 y_i)/n = 5.8038$$

$$res = 284.3434$$

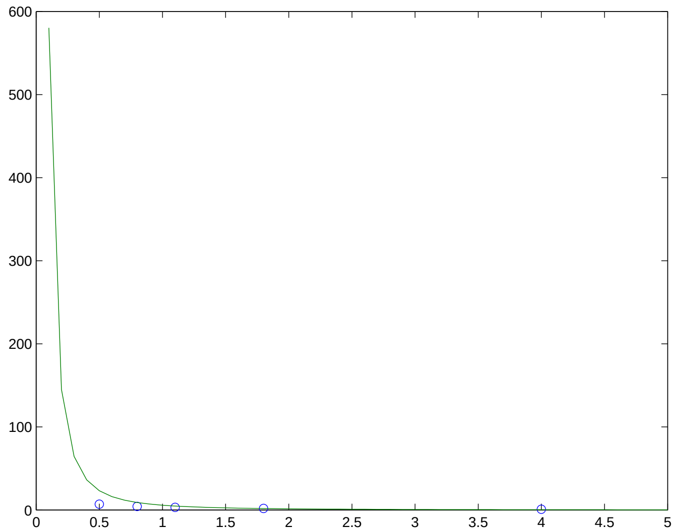

Figure 2: $y = b/x^2$

2 Problem 2

Using SVD Method

$$A = \begin{bmatrix} 4 & -2 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \\ 4 & -2 & 1 \end{bmatrix}, \text{ and B} = A^T A = \begin{bmatrix} 34 & 0 & 10 \\ 0 & 10 & 0 \\ 10 & 0 & 5 \end{bmatrix}$$

The next step is to find the eigenvectors and eigenvalues for B using the MATLAB command [V,S]=eig(B)

$$V = \begin{bmatrix} 0.2973 & 0 & -0.9548 \\ 0 & -1 & 0 \\ -0.9548 & 0 & 0.2973 \end{bmatrix}$$

where $\sigma_1^2 = 1.8861, \sigma_2^2 = 10, \sigma_3^2 = 37.1139$

The third step is to find U where $u_j = \sigma_j^{-1} A V_j$

$$U = \begin{bmatrix} 0.1707 & 0.6325 & 0.6757 \\ -0.4787 & 0.3162 & 0.2055 \\ -0.6952 & 0 & 0.0488 \\ -0.4787 & -0.3162 & 0.2055 \\ 0.1707 & -0.6352 & 0.6757 \end{bmatrix}$$

The coefficients of the quadratic equation can then be found

$$c = VS^{+}U^{T}Y = \begin{bmatrix} 0.9929 \\ 2.1300 \\ 3.1543 \end{bmatrix}$$

$$res = 0.1223$$

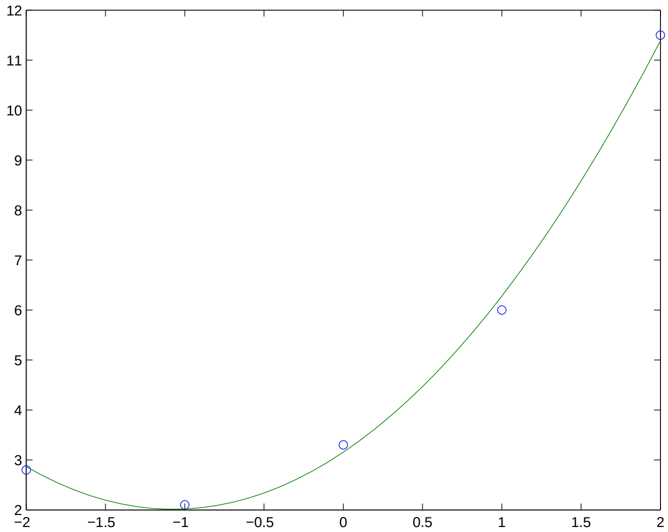

Figure 3: $y = ax^2 + bx + c$

3 Problem 3

3.1 Part 1

Using QR Factorization, to find the coefficients for the equation y = $ce^a x$ we will transform it to the linear equation ln(y) = ln(c) + ax, so we will solve for the equation relating ln(y) and x

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \\ 5 & 1 \end{bmatrix}$$

using the MATLAB command [Q,R] = qr(A,0) gives the results

$$Q = \begin{bmatrix} -0.1348 & -0.7628 \\ -0.2697 & -0.4767 \\ -0.4045 & -0.1907 \\ -0.5394 & 0.0953 \\ -0.6742 & 0.3814 \end{bmatrix}, R = \begin{bmatrix} -7.4162 & -2.0226 \\ 0 & 0.9535 \end{bmatrix}$$

The coefficients for the linear equations are

$$c = R^{-1}Q^TY = \begin{bmatrix} 0.7475 \\ -1.0123 \end{bmatrix}$$

so the non-linear equation is y = $0.3634e^{0.7475x}$

$$res = 0.1773$$

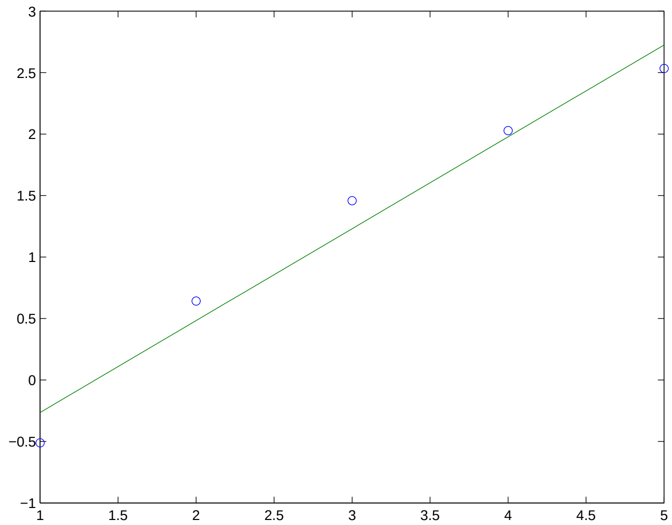

Figure 4: $ln(y) = ln(c) + ax$

3.2 Part 2

Using QR Factorization, to find the coefficients for the equation y = $ce^a x$ we will transform it to the linear equation ln(y) = ln(c) + a*ln(x), so we will solve for the equation relating ln(y) and x

$$A = \begin{bmatrix} 0 & 1 \\ 0.6931 & 1 \\ 1.0986 & 1 \\ 1.3863 & 1 \\ 1.6094 & 1 \end{bmatrix}$$

using the MATLAB command [Q,R] = qr(A,0) gives the results

$$Q = \begin{bmatrix} 0 & -0.8761 \\ -0.2784 & -0.4071 \\ -0.4412 & -0.1328 \\ -0.5563 & 0.0618 \\ -0.6464 & 0.2128 \end{bmatrix}, R = \begin{bmatrix} -2.4899 & -1.9228 \\ 0 & -1.14150 \end{bmatrix}$$

The coefficients for the linear equations are

$$c = R^{-1}Q^T Y = \begin{bmatrix} 1.8860 \\ -0.5755 \end{bmatrix}$$

so the non-linear equation is y = $0.5624x^{1.8860}$

$$res = 0.0193$$

The result is shown in figure 5.

3.3 Notes

1. If you used Gramm-Shmidt to obtain the QR factorization by hand the results will be different than MATLAB, the values in Q, and R are all the same but with different signs. However, the result should be the same at the end. The reason is that MATLAB seems to use another method than Gramm-Schmidt to obtain Q.

2. The residuals computed above are for the linearized equations.

4 Problem 4

Using Normalized Equations

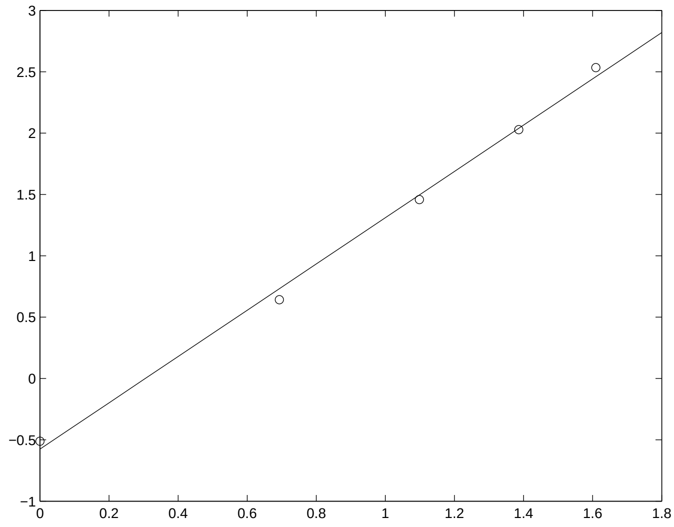

Figure 5: $ln(y) = ln(c) + aln(x)$

4.1 Linear Equation

The coefficients for the equation are

$$c = \begin{bmatrix} 4.1402 \\ -5.8261 \end{bmatrix}$$

$$res = 26.8235$$

4.2 Quadratic Equation

The coefficients for the equation are

$$c = \begin{bmatrix} 0.0772 \\ 3.3730 \\ -4.4649 \end{bmatrix}$$

$$res = 23.2930$$

4.3 Cubic Equation

The coefficients for the equation are

$$c = \begin{bmatrix} 0.0041 \\ 0.0116 \\ 3.6556 \\ -4.7186 \end{bmatrix}$$

$$res = 23.2217$$

4.4 Linear reciprocal

$$a = \sum_1^n (x_i y_i)/n$$

$$a = 109.9160$$

$$res = 5.4234e+004$$

The residual is very high because the values of x, and y are increasing and the linear reciprocal function is not suitable to fit these values.

4.5 Quadratic reciprocal

$$b = \sum_1^n (x_i^2 y_i)/n$$

$$b = 821.3122$$

$$res = 1.0860e+007$$

The residual is very high because the values of x, and y are increasing and the quadratic reciprocal function is not suitable to fit these values.

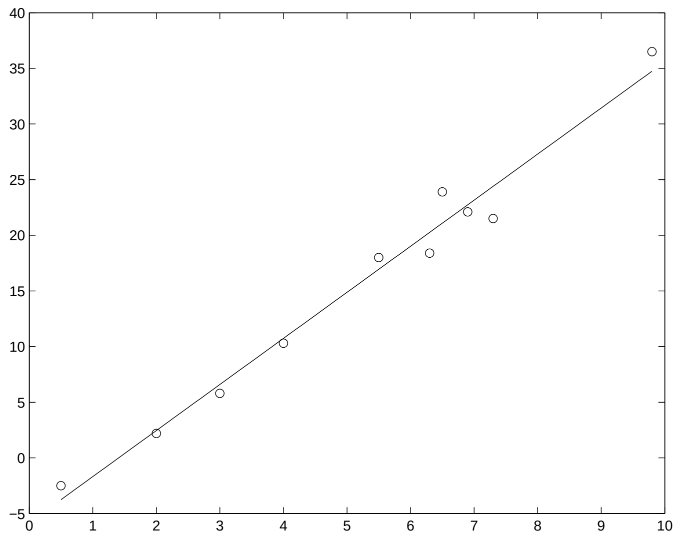

Figure 6: $y = ax + b$

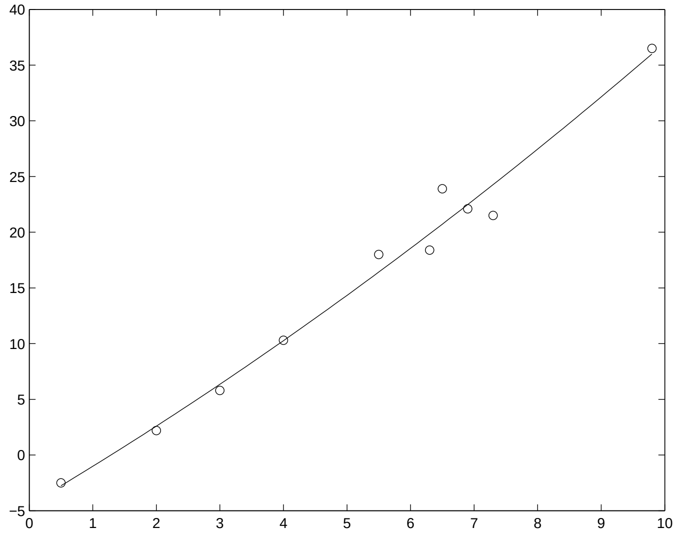

Figure 7: $y = ax^2 + bx + c$

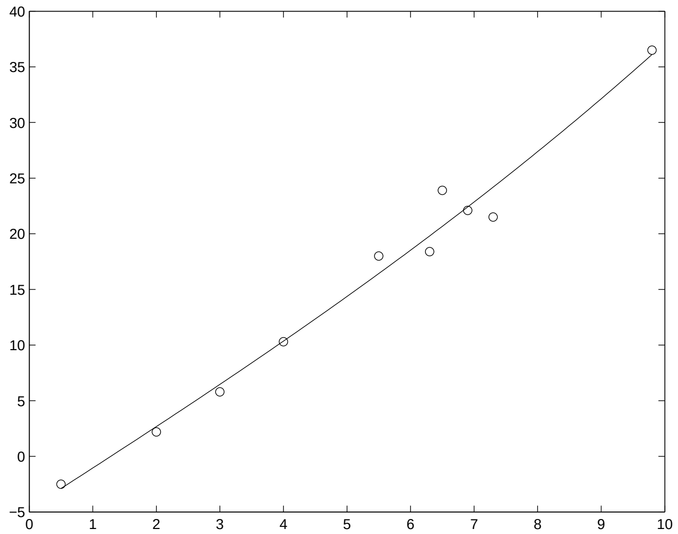

Figure 8: $y = ax^3 + bx^2 + cx + d$

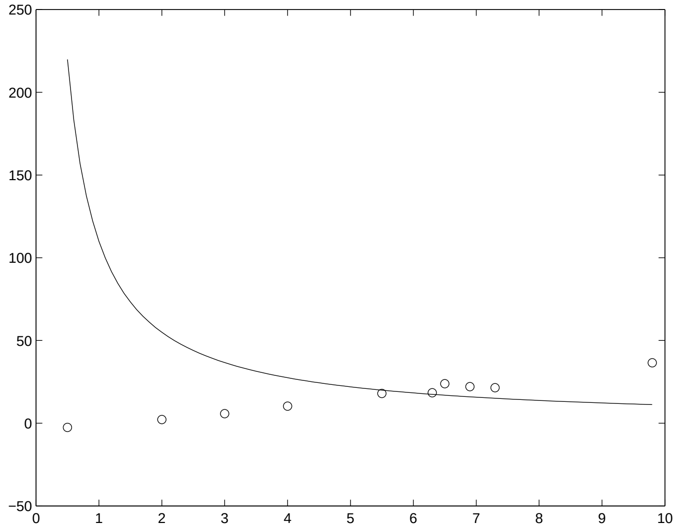

Figure 9: $y = a/x$

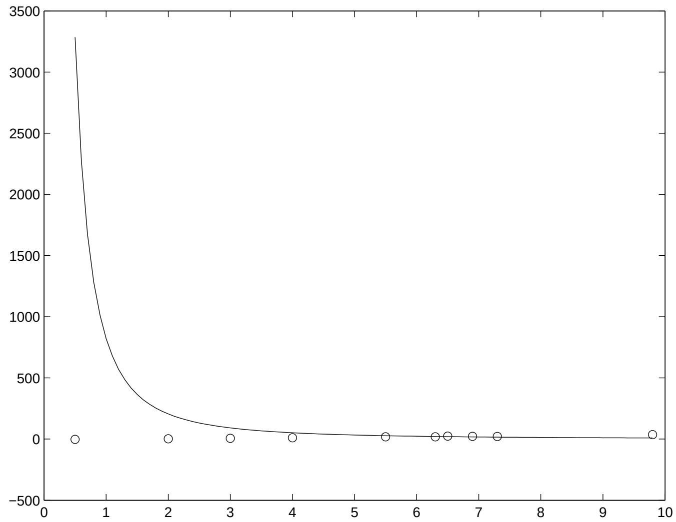

Figure 10: $y = b/x^2$