Unit VI - Epilogue: Linear Systems

- terminology of linear systems
- homogeneous linear systems
  - vector subspace connections
  - solution space and nullspace
- non-homogeneous linear systems
  - solution set is not a vector space
  - particular and general solutions
  - existence and uniqueness of solutions
  - under- and over-determined systems

Terminology

- an m×n *linear system* consists of
  - m simultaneous linear equations in...
  - n variables
- using matrices we can write Ax = b where
  - A m×n is the *coefficient matrix*
  - $x \in R^n$ is the vector of *unknowns*
  - $b \in R^m$ is the vector of *constants*
- a *solution* to the system is a vector x which satisfies Ax = b
  - a *particular* solution is any specific x which is a solution
  - the *general* solution is a vector x which gives the form of all solutions

Some more terminology

- a linear system is
  - *consistent* if there is at least one solution and *inconsistent* if there is no solution
  - *homogenous* if b = 0, i.e. Ax = 0 and *non-homogenous* if b ≠ 0
- the form of the coefficient matrix is used to describe the system
  - *triangular* if A is a triangular matrix
  - *square* if A is a square matrix
  - *echelon* if A is an echelon matrix
  - *diagonal* if A is a diagonal matrix
- the *augmented matrix* M = [A|b] in block matrix form consists of A with the vector of constants b as an additional column

Some stuff about solutions

- a homogenous system
  - always has the *trivial* solution x = 0....
  - may have non-trivial solutions as well, i.e. for which x≠0
- *equivalent systems* Ax=b have row-equivalent augmented matrices
- equivalent systems have identical solutions so...
- to solve a general system Ax=b we can convert it into an equivalent system which is easier to solve
- row-reduce the augmented matrix to...
  - an echelon form and use back-substitution to get the unknowns x [*Gaussian elimination*]
  - row canonical form and read off the variables from the entries corresponding to pivots [*Gauss-Jordan elimination*]

Solutions of a linear system

- a linear system Ax=b has
  - no solution OR...
  - a unique solution OR...
  - an infinite number of solutions
- WHY? [theorem 3.1 in the text]
  - suppose $Ax_1 = Ax_2 = b$ and $x_1 \neq x_2$
  - then $A(x_1 - x_2) = Ax_1 - Ax_2 = b - b = 0$
  - and for any scalar k we have
    $A[x_1 - k(x_1 - x_2)] = Ax_1 - k\,A(x_1 - x_2) = b - k\,0 = b$
  - so $x_1 - k(x_1 - x_2)$ is a solution of Ax = b for any scalar k
  - as long as there are an infinite number of scalars [e.g. for a real vector space] there will be an infinite number of solutions to Ax = b provided there are more than one

Homogenous system: general solution

- the set of solutions of a homogenous m×n linear system Ax=0 is a subspace of R^n: the *nullspace* of A
  - its dimension is nullity(A) = n - r where rank(A) = r
  - this is the number of parameters in the general solution
  - it's also the number of free variables in the echelon form of the row-reduced coefficient matrix [see text p.134 & p.179]
- the general solution of Ax=0 is any vector x_g in the nullspace of A
  - if $\{v_1, ..., v_k\}$ is a basis for the nullspace of A ....
  - x_g can be written as a linear combination
    $x_g = c_1 v_1 + c_2 v_2 + ... + c_k v_k$

Non-homogeneous system: solution set

- what about a non-homogenous system Ax=b?
  - it's solution set is definitely not a subspace of R^n
  - e.g. $A(2x_1) = 2Ax_1 = 2b \neq b$ in general so the set is not closed under scalar multiples
  - in fact the zero vector is not in the solution set of Ax=b [see text p.123]
- the *associated homogenous system* is Ax=0
  - the one with the same coefficient matrix as Ax=b
- we can find the general solution of Ax=b using
  - a particular solution x_p of Ax=b and....
  - the general solution x_g of the associated homogeneous system Ax=0

Non-homogenous system: general solution

- WHY?
  - suppose $Ax_p = b$ gives some particular solution
  - let x be any [i.e. the general] solution of Ax=b
  - then $A(x-x_p) = Ax - Ax_p = b - b = 0$
  - so $x-x_p$ is in the nullspace of A so $x - x_p = c_1v_1 + c_2v_2 + ... + c_kv_k$
  - solve for $x = x_p + [c_1v_1 + c_2v_2 + ... + c_kv_k] = x_p + x_g$ as required
  - now to go the other way .... suppose $x = x_p + x_g = x_p + c_1v_1 + c_2v_2 + ... + c_kv_k$
  - then $Ax = A[x_p + c_1v_1 + c_2v_2 + ... + c_kv_k]$
    $= Ax_p + c_1Av_1 + c_2Av_2 + ... + c_kAv_k$
    $= b + 0 + ... + 0$
    $= b$

General and particular solution

So the general solution of Ax=b is of the form

$$x = x_p + x_g$$

a particular solution of Ax=b (x_p)

the general solution of the associated homogenous system Ax=0 (x_g)

Example: a non-homogeneous system

[problem 3.12] Find the general solutions of the systems

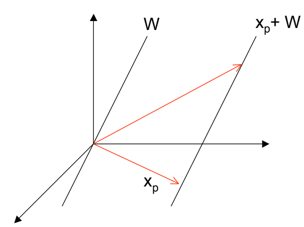
$$\begin{aligned} x_1 - 3x_2 + 2x_3 - x_4 + 2x_5 &= 2 \\ 3x_1 - 9x_2 + 7x_3 - x_4 + 3x_5 &= 7 \\ 2x_1 - 6x_2 + 7x_3 + 4x_4 - 5x_5 &= 7 \end{aligned}$$

$$\begin{aligned} x_1 + 2x_2 - 3x_3 + 4x_4 &= 2 \\ 2x_1 + 5x_2 - 2x_3 + x_4 &= 1 \\ 5x_1 + 12x_2 - 7x_3 + 6x_4 &= 3 \end{aligned}$$

Example: a non-homogeneous system (cont.)

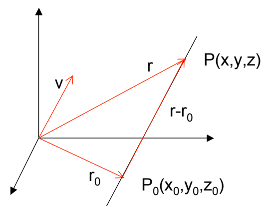
Geometric view in R^3

- the general solution of Ax=0 is a subspace W of R^3, e.g. a line through the origin
- the general solution of Ax=b, b≠0, is NOT a subspace of R^3, e.g. it's a line not through the origin



Application: equation of a line in R^3

- the line is defined by
  - a point on the line: P_0 with position vector r_0 and...
  - direction vector v = (a,b,c)



Application: equation of a line in R^3

- a general point P with position vector r=(x,y,z) gives a vector $r-r_0$ parallel to v
- so the required relationship is $r-r_0 = tv$ [where t is any scalar] or ... $r = r_0 + tv$
- solving for t by coordinates gives the *parametric equations* of the line

$$(t=)\,\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}$$

- suitable adjustments are needed if a,b, or c = 0
- the same equation form applies in R^n

Example: equation of a line in R^3

[problem 1.22] Find the parametric equations of the line (a) through points P(1,3,2) and Q(2,5,-6), and (b) containing the point P(1,-2,4) and perpendicular to the plane with equation 3x+5y+7z = 15.

How to solve a non-homogenous system

- to solve Ax=b if A is **square** and **non-singular** you can
  - invert A and ....
  - multiply both sides of Ax = b on the left by A^{-1} to get ...
  - the unique solution $x = A^{-1}b$
- equivalent conditions for a square system Ax = b are:
  - A is invertible [or, equivalently, non-singular]
  - the system is consistent for any vector b
  - there is a unique solution for any vector b

How to solve a non-homogenous system

- to solve Ax=b if A is **not square** and/or **singular** you can use *Gaussian elimination*
  - write the augmented matrix [A | b]
  - row reduce it to echelon form
  - back substitute to solve for the pivots
  - the free variables are arbitrary constants in the general solution
- this method is the best practical one even for non-singular square systems

Non-homogeneous systems

- an $m\times n$ system Ax=b with augmented matrix M is consistent if and only if
  - rank A = rank M
  - b is in the column space of A
- a consistent $m\times n$ system Ax=b has
  - a unique solution if and only if rank A = n
  - an infinite number of solutions if rank A < n
- an $m\times n$ system Ax=b is consistent for all choices of b if and only if
  - the columns of A span R^m
  - rank A = m

Example: a non-homogeneous system

[problem 3.14] For the system

$$\begin{array}{rcrcrcl} x & + & 2y & + & z & = & 3 \\ & & ay & + & 5z & = & 10 \\ 2x & + & 7y & + & az & = & b \end{array}$$

(a) Find all values of *a* so there is a unique solution.

(b) Find all (*a*,*b*) so there is more than one solution and find the general solution.

Example: a non-homogeneous system (cont.)

Non-homogeneous systems

- an *under-determined system* has less equations than unknowns [m < n]
- an *over-determined system* has more equations than unknowns [m > n]

Non-homogeneous systems

- a consistent under-determined system
  - always has an infinite number of solutions [rank A ≤ m < n]
- a consistent over-determined system may have
  - a unique solution [rank A = n] or...
  - an infinite number of solutions [rank A < n]
- in general an over-determined system
  - cannot be consistent for all b [because n (< m) columns cannot span all of R^m]
  - often results from a linear fit of experimental data ...leading to the idea of 'best-fit' .... least-squares fitting .... SyDe 312 .... **see you in 3A**