Unit IV - Diagonalization

- diagonalizing matrices
- characteristic polynomial
- eigenvalues, eigenvectors and eigenspaces
- similarity of matrices or linear operators

Illustrative example

- find the matrix representation of the linear operator T: $R^2 \to R^2$ defined by T(x,y) = (3x+y, x+3y) with respect to the basis β = {(1,1), (1,-1)}
- first calculate the coordinates of any vector with respect to β: (a,b) = $[(a+b)/2, (a-b)/2]_\beta$
- now apply this to the images of the basis vectors:
  - T(1,1) = (4,4) = $[4,0]_\beta$
  - T(1,-1) = (2,-2) = $[0,2]_\beta$
- so the required matrix representation is a diagonal matrix....

$$[T]_\beta = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$$

...Illustrative example

- an alternative approach to get the matrix...
- the change of basis matrix and its inverse are

$$P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

- then the required matrix representation is

$$P^{-1}AP = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$$

- diagonal matrices have very convenient properties for calculations, so it's important if we can represent a linear operator this way

Diagonalization

- not all linear operators can be represented by diagonal matrices with respect to some basis
- a *diagonalizable* matrix A has some [invertible] P so that $P^{-1}AP = D$ is diagonal
- equivalently a *diagonalizable* linear operator is one which can be represented by a diagonal matrix with respect to some basis
- our shopping list:
  - which matrices can be diagonalized...
  - how to find the appropriate basis...and
  - how to calculate the diagonal matrix

Finding the right basis

- linear operator T: V $\to$V
- $\beta = \{u_1, u_2, \ldots ,u_n\}$ a basis of V
- the matrix representation $[T]_\beta$ is diagonal if and only if

  $[T(u_1)]_\beta = [\lambda_1, 0, 0, \ldots, 0]_\beta$

  $[T(u_2)]_\beta = [0, \lambda_2, 0, \ldots, 0]_\beta$

  . . . .

  $[T(u_n)]_\beta = [0, 0, \ldots, 0, \lambda_n]_\beta$

- conclusion.... a linear operator T: V $\to$V is diagonalizable if and only if there is a basis of V $\beta = \{u_1, u_2, \ldots ,u_n\}$ so that **$T(u_i) = \lambda_i u_i$**

Eigenvectors and eigenvalues

- a vector v$\in$V for which T(v) = λv is called an *eigenvector* of T with *eigenvalue* [scalar] λ
- so we've seen that
  - T is diagonalizable if and only if there is a basis of V consisting of eigenvectors of T and....
  - the diagonal entries of the matrix representation with respect to that basis are simply the corresponding eigenvalues
- geometrically the transformation from v to T(v) leaves the direction of an eigenvector unchanged and its length magnified by λ

  WARNING: *this is not the same thing as a dilation or contraction map [why?]*

Finding eigenvectors and eigenvalues

- we work with matrices now for simplicity
  - for a linear operator T_A we can use say the matrix representation A with respect to the standard basis
  - all these concepts and calculations are relevant to A
  - A must be square in all of this of course
- an eigenvector v of A satisfies $Av = \lambda v$ for some λ
- equivalently we have the matrix equation

$$[A - \lambda I]\, v = 0$$

- v = 0 is obviously a possible solution, but not very interesting
  - the zero vector is technically an eigenvector of any matrix since $A0 = \lambda 0$ for any λ
- what about non-zero solutions?

Finding eigenvectors and eigenvalues

- a non-zero solution of $[A - \lambda I]\, v = 0$ exists if and only if the matrix $A - \lambda I$ is not invertible, i.e. $A - \lambda I$ must be a singular matrix
  - otherwise we could invert $A - \lambda I$ and get the unique solution $v = [A - \lambda I]^{-1}0 = 0$, i.e. only the zero solution
- equivalently we have non-zero eigenvectors if and only if the rank of $A - \lambda I < n$ .....or
- equivalently we want **$\det(A - \lambda I) = 0$**
- to find this you simply subtract λ from each diagonal entry of A and take the determinant

Finding eigenvectors and eigenvalues

- this equation $\det(A - \lambda I) = 0$ is called the *characteristic equation* of the matrix A
  - it's a polynomial of degree n if A is $n \times n$
  - its solutions give all the eigenvalues λ
- the number of times a root λ_i is repeated [$(\lambda - \lambda_i)^k$ is a repeated factor k times] is called the *algebraic multiplicity* of the eigenvalue λ_i
- once we know the $\lambda_1, \lambda_2, \lambda_3$,..... [solve for them] we take each one in turn and find the corresponding eigenvector(s) v by solving the linear system

$$[A - \lambda_i I]\, v = 0$$

Illustrative example repeated

- suppose we didn't know the values for the matrix in this illustrative example.... let's find the solution from scratch

Eigenspaces

- if v is an eigenvector then so is any multiple kv with the same eigenvalue:

  $A(kv) = k(Av) = k(\lambda v) = \lambda(kv)$
- if v and w are eigenvectors for the same eigenvalue λ then so is v+w:

  $A(v+w) = Av+Aw = \lambda v+\lambda w = \lambda(v+w)$
- so for each eigenvalue λ the corresponding eigenvectors span a subspace E_λ, called the *eigenspace* of the eigenvalue λ
- the dimension of the eigenspace is called the *geometric multiplicity* of the eigenvalue λ
- **a complete solution consists of finding a basis of eigenvectors for each eigenspace (eigenvalue)**

Diagonalization

- what is the matrix P which diagonalizes a matrix A by providing $D = P^{-1}AP$?
  - P is the change of basis matrix between the standard basis and the special basis consisting of eigenvectors of A
  - so P consists of columns which are the eigenvectors expressed in standard coordinates
- D is very useful for simplifying calculations of powers of A:
  - A can be factorized as $A = PDP^{-1}$
  - then $A^2 = (PDP^{-1})(PDP^{-1}) = PD(P^{-1}P)DP^{-1} = PD^2P^{-1}$....
  - $A^n = (PDP^{-1})....(PDP^{-1}) = PD^nP^{-1}$
  - powers of a diagonal matrix are trivial to calculate
- these comments apply to calculation of polynomials of matrices too

Example: eigenvalues and eigenvectors

[problem 9.9] (a) Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 3 & -4 \\ 2 & -6 \end{bmatrix}$ (b) diagonalize A if possible (c) find A^6.

Example: eigenvalues and eigenvectors

[problem 9.11] (a) Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 5 & -1 \\ 1 & 3 \end{bmatrix}$ (b) diagonalize A if possible.

IMPORTANT stuff about eigenvalues

- only square matrices have eigenvalues
- an n×n matrix has at most n distinct eigenvalues [why?]
- **eigenvectors corresponding to distinct eigenvalues are always linearly independent** [see problem 9.21]
- **the geometric multiplicity of an eigenvalue never exceeds its algebraic multiplicity**
- if λ is an eigenvalue of an invertible matrix A then $1/\lambda$ is an eigenvalue of A^{-1}

Example: eigenvalues and eigenvectors

[problem 9.14] (a) Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{bmatrix}$ (b) diagonalize A if possible.

Example: degenerate eigenvalue

[problem 9.15] (a) Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 3 & -1 & 1 \\ 7 & -5 & 1 \\ 6 & -6 & 2 \end{bmatrix}$ (b) diagonalize A if possible.