Unit II - Matrix arithmetic

- matrix multiplication
- matrix inverses
  - elementary matrices
  - finding the inverse of a matrix
- determinants

Things we can already do with matrices

- equality A = B
- sum A + B
  - additive identity the 0 matrix
  - additive inverse -A
- scalar product cA
- transpose A^T
- symmetric and skew-symmetric matrices
- links between matrix spaces and spanning sets, bases etc.
- these are matrix vector space concepts
- unlike for vector spaces we introduce a matrix product...the true motivation for this will not emerge until the next unit [**linear mappings**]

Defining the matrix product

- if #columns of A = #rows of B then we can define the matrix product AB
- if $A=(a_{ij})$ is m×p and $B=(b_{ij})$ is p×n...
- the product AB is m×n and defined by

$$\mathbf{(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{ik}b_{kj}}$$

- the ijth entry of AB is formed by multiplying together corresponding entries in row i of A and column j of B and adding the k products together
- in general **AB ≠ BA**
  - both products are only defined if both matrices are square
  - even if both are square the product is not commutative [most examples you try probably will fail]

Things that fail in matrix arithmetic

- AB ≠ BA even if both are defined
- AB = 0 does NOT imply either A = 0 or B = 0
- no cancellation rule: AB = AC does NOT imply that B = C
- A^2 = A does NOT imply that A = I
- in general there is no multiplicative inverse B for a matrix A, i.e. a B so that AB = I, even if A is square so that B may be possible

Example: AB≠BA & AB=0 with A,B≠0

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -3 & 2 & -1 \\ -2 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & -1 & 1 \\ -3 & 2 & -1 \\ -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ -3 & 2 & -1 \\ -2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -11 & 6 & -1 \\ -22 & 12 & -2 \\ -11 & 6 & -1 \end{bmatrix}$$

Example: AB=AC with B≠C

$$A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & -3 \\ 4 & -3 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 4 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 1 & -1 & -2 \\ 3 & -2 & -1 & -1 \\ 2 & -5 & -1 & 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & -3 \\ 4 & -3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} -3 & -3 & 0 & 1 \\ 1 & 15 & 0 & -5 \\ -3 & 15 & 0 & -5 \end{bmatrix}$$

$$AC = \begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & -3 \\ 4 & -3 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 & -2 \\ 3 & -2 & -1 & -1 \\ 2 & -5 & -1 & 0 \end{bmatrix} = \begin{bmatrix} -3 & -3 & 0 & 1 \\ 1 & 15 & 0 & -5 \\ -3 & 15 & 0 & -5 \end{bmatrix}$$

Example: $A^2=A$ with $A \neq I$

$$A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$

- a matrix for which $A^2=A$ is called *idempotent*

Powers and polynomials of square matrices

- for a square matrix A we define *powers*
  - $A^2 = AA$
  - $A^3 = A^2A$
  - ...
  - $A^n = A^{n-1}A$
- how do we know that these are well-defined?
- ...because matrix multiplication is not totally pathological and satisfies the associative law:
  - $A(BC) = (AB)C$
  - [see text problem 2.10] provided the triple product is defined [right sizes of matrices]
- we can also calculate polynomials of matrices by combining powers and vector space operations....

Example: Polynomials of a matrix

- calculate $f(A) = 2A^3 - 4A + 5$ for the given A

$$A = \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix} = \begin{bmatrix} 9 & -4 \\ -8 & 17 \end{bmatrix}$$

$$A^3 = AA^2 = \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} 9 & -4 \\ -8 & 17 \end{bmatrix} = \begin{bmatrix} -7 & 30 \\ 60 & -67 \end{bmatrix}$$

$$f(A) = 2\begin{bmatrix} -7 & 30 \\ 60 & -67 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -13 & 52 \\ 104 & -117 \end{bmatrix}$$

Some useful properties

- matrix transpose
  - ✓ $(A^T)^T = A$
  - ✓ $(kA)^T = kA^T$
  - ✓ $A^T + B^T = (A+B)^T$
  - ✓ **$(AB)^T = B^T A^T$**
- distributive laws
  - $A(B+C) = AB + AC$
  - $(A+B)C = AC + BC$
- zero product: $0A = 0$ and $B0 = 0$
- associative law
  - $(AB)C = A(BC)$ and special case...
  - $k(AB) = (kA)B = A(kB)$ k a scalar

Identity matrix

- the (multiplicative) *identity matrix* is I, or I_n to emphasize the size

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

- the identity is:
  - two-sided for square matrices: $IA = AI = A$
  - left and right for an m×n matrix: $I_mB = BI_n = B$
- note that $(kI)A = k(IA) = kA$

Matrix inverse

- A is a square matrix
- a matrix A^{-1} so that $AA^{-1} = A^{-1}A = I$ is called an *inverse* for A
- not all square matrices have inverses
- those that do are called *invertible* or *non-singular*
- all of A, A^{-1}, and I are the same square size above
- if A is non-singular the inverse A^{-1} is unique
  - say B and C are both inverses of A
  - then $B = BI = B(AC) = (BA)C = IC = C$
  - this works because the inverse is a 2-sided inverse

Properties of matrix inverse

- A invertible if and only
  - A^{-1} invertible ...... $(A^{-1})^{-1} = A$
  - A^n invertible ...... $(A^n)^{-1} = (A^{-1})^n$
  - kA invertible ...... $(kA)^{-1} = (1/k)A^{-1}$
  - A^T invertible ...... $(A^T)^{-1} = (A^{-1})^T$
- if A, B invertible and the same size then
  - AB is invertible and **$(AB)^{-1} = B^{-1}A^{-1}$**
- a diagonal matrix D is invertible if and only if no diagonal entry is zero ......

$$\begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}^{-1} = \begin{bmatrix} 1/d_1 & 0 & \cdots & 0 \\ 0 & 1/d_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1/d_n \end{bmatrix}$$

Why define matrix product as we did?

- matrix multiplication isn't very intuitive in the raw
- the true motivation for our definition will be seen when we study **linear mappings** in the next unit
  - a matrix can be used to represent a linear mapping
  - the matrix product AB is then the result of doing two of these mappings in succession
- we'll learn how to do calculations with the matrix product before drawing this linear mapping connection
- two simple illustrations of matrix product:
  1. writing systems of equations
  2. representing the effect of an elementary row operation by a matrix product

1. System of equations

- the system of equations...

$$\begin{aligned} x - y + z &= 4 \\ 3x - 2y + 2w &= 2 \\ -4x + y + 11z &= -1 \\ -x + 3z - w &= -5 \end{aligned}$$

- ...can be written as a matrix product, considering the column vector u = $[x\ y\ z\ w]^T$ as a 4×1 matrix

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 3 & -2 & 0 & 2 \\ -4 & 1 & 11 & 0 \\ -1 & 0 & 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -1 \\ -5 \end{bmatrix}$$

- or simply Au = b
- A is called the *coefficient matrix* of the system

System of equations

- a linear system Au = b is an equation involving matrix products
- if the coefficient matrix A is invertible and we know A^{-1} we can use it to calculate *THE* solution vector u:

  $A^{-1}A\ u = Iu = u = A^{-1}b$
- this method fails if A is
  - not a square matrix [e.g. the system below] or
  - square, but non-invertible [e.g. the A in the previous slide]

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 3 & -2 & 0 & 2 \\ -4 & 1 & 11 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}$$

2. Elementary row operations

- we can represent the effect of an elementary row operation on a matrix A by a matrix product EA
  - E is called an *elementary matrix*
  - multiplying A on the left by E produces a new matrix EA with the row operation applied to A
- apply a chain of row operations to both sides of a system of equations Ax = b:

  $E_1A = E_1b \rightarrow E_2E_1A = E_2E_1b \rightarrow E_3E_2E_1A = E_3E_2E_1b$ ....
- the elimination method is based on:
  - applying the same operations to the b vector and
  - obtaining a product on the left hand side in echelon form
- note that the matrices multiply in the reverse order of doing the operations [inside out]

Three types of elementary matrices

multiply row 2 by c:

$$\begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ ca_{21} & ca_{22} \end{bmatrix}$$

interchange rows 2&3:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

add row 3 to row 1:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11}+a_{31} & a_{12}+a_{32} & a_{13}+a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Elementary matrices

- **an elementary matrix E can be obtained by applying the required row transformation to the identity matrix I**
- elementary matrices are invertible

$$\begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1/c \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- so a product of elementary matrices is invertible ......a sequence of row operations can be undone

Row canonical form of a matrix

- a matrix A is in *row-canonical form* if
  - it is in echelon form
  - each pivot entry is 1
  - all entries above a pivot entry are zero
- this is a more-specialized form of echelon matrix
- any matrix is row equivalent to a ***unique*** matrix in row-canonical form

Combining three concepts

- for an n×n matrix A the following are equivalent:
  - A is invertible
  - A is a product of elementary matrices
  - the row canonical matrix of A is I_n
- you should study the proof of this result in the text problem 3.35
- this provides a convenient method of calculating the inverse of a matrix....

Finding matrix inverses

- suppose $A = E_1E_2....E_k$
- write $E_1^{-1}E_1 = I$ then....
- $E_2^{-1}(E_1^{-1}E_1)E_2 = I$ and so on, until....
- $E_k^{-1}(E_{k-1}^{-1}.... (E_2^{-1}(E_1^{-1}E_1)E_2)....E_{k-1})E_k = I$
- re-arrange as $[E_k^{-1}E_{k-1}^{-1}.... E_2^{-1}E_1^{-1}] [E_1E_2....E_{k-1}E_k] = I$
- or $[E_k^{-1}E_{k-1}^{-1}.... E_2^{-1}E_1^{-1}] [A] = I$
- so $A^{-1}= E_k^{-1}E_{k-1}^{-1}.... E_2^{-1}E_1^{-1}$
- since A^{-1} is unique this product is the required inverse matrix of A

A convenient practical method

- arrange A and *I* side by side: [A, *I*]
- reduce A to *I* and apply the same row reduction steps to *I* simultaneously
- the result is $[I, A^{-1}]$
- example find the inverse of A = $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 3 & 0 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & -3 & 3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

- the inverse is $A^{-1} = \begin{bmatrix} 7 & -3 & 3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

Example: Text 3.32 (a)

Find the inverses [if possible] of the following matrices:

$$A = \begin{bmatrix} 1 & 2 & -4 \\ -1 & -1 & 5 \\ 2 & 7 & -3 \end{bmatrix}$$

...Example: Text 3.32 (b)

$$B = \begin{bmatrix} 1 & 3 & -4 \\ 1 & 5 & -1 \\ 3 & 13 & -6 \end{bmatrix}$$

Determinants

- we define a useful concept for a SQUARE matrix A of any size: the *determinant* of A is a real number
- first det [a] = a
- next: $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$
- then: $\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}[a_{22}a_{33} - a_{23}a_{32}] - a_{12}[a_{21}a_{33} - a_{23}a_{31}] + a_{13}[a_{21}a_{32} - a_{22}a_{31}]$$

Examples: determinants

$$\begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix} = 3(-2) - 1(4) = -10$$

$$\begin{vmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{vmatrix} = 1\begin{vmatrix} 5 & 6 \\ -8 & 9 \end{vmatrix} - 2\begin{vmatrix} -4 & 6 \\ 7 & 9 \end{vmatrix} + 3\begin{vmatrix} -4 & 5 \\ 7 & -8 \end{vmatrix}$$

$$= 1(45-(-48)) - 2(-36-42) + 3(32-35) = 240$$

$$\begin{vmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{vmatrix} = -2\begin{vmatrix} -4 & 6 \\ 7 & 9 \end{vmatrix} + 5\begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} - (-8)\begin{vmatrix} 1 & 3 \\ -4 & 6 \end{vmatrix}$$

$$= -2(-36-42) + 5(9-21) + 8(6-(-12)) = 240$$

Alternative expansions

- you can expand a determinant along *any* row or column, e.g. the previous example:

$$\begin{vmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{vmatrix} = -2\begin{vmatrix} -4 & 6 \\ 7 & 9 \end{vmatrix} + 5\begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} - (-8)\begin{vmatrix} 1 & 3 \\ -4 & 6 \end{vmatrix}$$

$$= -2(-36-42) + 5(9-21) + 8(6-(-12)) = 240$$

- include the appropriate sign $(-1)^{i+j}$ [see cofactors]
- so choose the way that gives the easiest arithmetic, especially when there are zero entries
- a simple trick also works for 3×3 determinants...

Easy trick for 3×3 determinants

- a 3×3 det consists of:
  - six products with one entry from each row
  - signed with ±1 according to 'direction'

$$\begin{vmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{vmatrix} = 1(5)(9) + 2(6)(7) + 3(-4)(-8)$$ left→right

$$- 1(6)(-8) - 2(-4)(9) - 3(5)(7)$$ right→left

- **warning....** this method only works for 3×3 determinants!

Some useful stuff about determinants

- if A is an n×n (square) matrix
  - det (A) = det (A^T)
  - det A = 0 if A has a row or column of zeros
  - if A is triangular det (A) = $a_{11}a_{22}.....a_{nn}$
  - **det (AB) = det (A) det (B)**
  - det (A^{-1}) = 1/det (A)
  - det (kA) = k^n det (A)
  - det (A+B) ≠ det (A) + det (B) in general

Using row ops to evaluate determinants

- if E is an elementary matrix
  - det (E) = c if E multiplies a row by c
  - det (E) = -1 if E interchanges two rows
  - det (E) = 1 if E adds a multiple of one row to another
- so elementary row ops change a determinant in general
- so do elementary column ops
- but all these changes are predictable and can be kept track of...

Using row ops to evaluate determinants

- direct evaluation of a determinant is usually not feasible except for small matrices, so...
- apply row or column operations to a determinant to obtain a row or column with a single non-zero entry
- then expand the determinant about that entry
- obtain a determinant of size one less
- **AVOID POTENTIAL STUDENT ERRORS:**
  - multiplying a row by k multiplies the det by k
  - interchanging two rows changes the sign of the det
  - when you add a multiple of a row to another be sure to put it in the proper place, or else you have changed the det

Examples: using row ops for determinants

$$\begin{vmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 13 & 18 \\ 0 & -22 & -12 \end{vmatrix} = -2 \begin{vmatrix} 1 & 2 & 3 \\ 0 & 13 & 18 \\ 0 & 11 & 6 \end{vmatrix}$$

$$= (-2) \begin{vmatrix} 13 & 18 \\ 11 & 6 \end{vmatrix} (-2)(78 - 198) = 240$$

- the amount of work involved for a 3×3 determinant is about the same as for a direct evaluation
- for a 4×4 determinant or bigger the ONLY sensible approach is to simplify with row or column ops...

Example: Text 8.4(a)

$$\begin{vmatrix} 2 & 5 & -3 & -2 \\ -2 & -3 & 2 & -5 \\ 1 & 3 & -2 & 2 \\ -1 & -6 & 4 & 3 \end{vmatrix}$$

Example: Text 8.4(b)

$$\begin{vmatrix} 6 & 2 & 1 & 0 & 5 \\ 2 & 1 & 1 & -2 & 1 \\ 1 & 1 & 2 & -2 & 3 \\ 3 & 0 & 2 & 3 & -1 \\ -1 & -1 & -3 & 4 & 2 \end{vmatrix}$$

Minors and cofactors

- let A = (a_{ij}) be a n×n matrix
- the sub-determinant $|M_{ij}|$ obtained by deleting row i and column j from |A| is called the *minor* of a_{ij}
- the signed minor $A_{ij} = (-1)^{i+j}|M_{ij}|$ is called the *cofactor* of a_{ij}
- the transpose of the (n×n) matrix of cofactors of A is called the (classical) *adjoint* of A:

  adj (A) = $[A_{ij}]^T$
- this provides a [not very useful but famous] formula for the inverse matrix:

  A^{-1} = (1/det A) adj (A)
- the method is mainly of theoretical interest

Example: Text 8.6

Find the inverse of

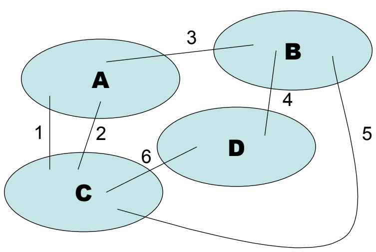
$$\begin{bmatrix} 1 & 3 & 3 \\ 2 & 3 & 4 \\ 5 & 8 & 9 \end{bmatrix}$$

Applications of determinants?

- brute force numerical calculation is problematic
  - sore wrist syndrome (by hand)
  - roundoff error and overflow (machine)
  - so...always use row operations to evaluate determinants bigger than 3×3
- determinants are mainly useful to mathematicians for theoretical work
- there are some routine day-to-day applications in engineering math, e.g.
  - calculating the cross product of vectors in R^3
- I have minimized the topic because determinants really have few genuine engineering applications
- however you do need to know the basics as given

Incidence matrices

- consider a system with discrete components that interact at a finite number of well-defined points



Incidence matrices

- we can define the incidence matrix of this system to represent these interactions

	1	2	3	4	5	6
A	1	1	1	0	0	0
B	0	0	1	1	1	0
C	1	1	0	0	1	1
D	0	0	0	1	0	1

- rows are components, columns are interactions
- entries are {0,1}
- each column has exactly two non-zero entries
- these are coordinates of vectors in the binary vector space defined on the set of interactions {1,2,3,4,5,6}

Incidence matrices

- the vectors represent sets of interactions which can be deleted to isolate each component
- we can reduce the matrix to...

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

  and get a linearly independent set: {**1**23,**3**56,**4**6}
- the others can be expressed in terms of these three:
  - 1256 = 123+356
  - 12346 = 123 + 46
  - 345 = 356 + 46
  - 1245 = 123+356+46

Connecting various concepts

- for an n×n matrix A all these conditions are equivalent :
  - A is invertible
  - det (A) ≠ 0
  - rank A = n
  - the rows of A are a basis for the row space
  - the columns of A are a basis for the column space
  - the linear system Ax = b has a unique solution for any b: $x = A^{-1}b$
  - the echelon form for A has no zero row
  - the row canonical form for A is I_n
  - A is a product of elementary matrices