Unit I
Vector Spaces

Outline

- definition and examples
- subspaces and more examples
- ***three key concepts***
  - ***linear combinations and span***
  - ***linear independence***
  - ***bases and dimension***
- sums and direct sums of vector spaces

Vector spaces

- need two mathematical objects....
- set V of things called *vectors* u,v,w...∈ V
- set F of numbers called *scalars* k,l,m...∈ F
  - a *real* vector space uses real scalars R
  - a *complex* vector space uses complex scalars C
  - you may occasionally see *binary* vector spaces for which scalars can be {0,1}
- most (but not all) of the work here will involve real vector spaces

Vector space definition

- V is a *vector space over* F if useful algebraic operations are defined which satisfy various rules
  - the operations are modelled on the customary ones for operating with vectors in Euclidean space
  - the rules are abstracted from observations about useful properties for vectors in Euclidean space
  - the rules are reduced to a (minimal) set of axioms which are required to model the behaviour of vectors in Euclidean space
- **the vectors in V can be ANY mathematical objects that provide a convenient and useful structure so don't get too caught up in the geometric motivation**

Vector addition axioms

For any u,v ∈ V the *vector sum* u+v is defined and satisfies for all u,v,w ∈ V:

1. u+v ∈ V [closure]
2. u+v = v+u [commutative]
3. u+(v+w) = (u+v)+w [associative]
4. there is an additive identity 0 ∈ V so that u+0 = u [zero vector]
5. there is a vector -u ∈ V so that u+(-u) = 0 [additive inverse]

Scalar multiplication axioms

For any u ∈ V and k ∈ F the scalar multiple ku is defined and satisfies for all u,v ∈ V and k,l ∈ F :

6. ku ∈ V [closure]
7. k(u+v) = ku+kv [vector sum distributive]
8. (k+l)u = ku+lu [scalar sum distributive]
9. k(lu) = (kl)u [scalar multiplication associative]
10. 1u = u [an odd one but necessary to connect the two operations]

Re-cap

- A *vector space* involves
  - a set of vectors V
  - a vector sum to make new vectors (u+v)
  - a set of scalars F
  - a scalar multiple to make new vectors (kv)
  - various rules which are necessary if we want these operations to behave like vectors in Euclidean space
- as always the rules are reduced to a minimal set of axioms (**memorize them***)

* Hint: pay particular attention to red stuff

Example: zero vector space

- any vector space has to have at least one vector, i.e. the zero vector [see axiom 4]
- the smallest vector space is the *zero space* {0}
- 0+0 = 0

Example: Euclidean spaces

- F^n = the set of all n-tuples of elements in F
- real space R^n, complex space C^n
- vector space operations are defined as usual coordinate-wise:
  $(a_1,a_2,\ldots,a_n) + (b_1,b_2,\ldots,b_n) = (a_1+b_1, a_2+b_2,\ldots, a_n+b_n)$
  $k(a_1,a_2,\ldots,a_n) = (ka_1,ka_2,\ldots,ka_n)$
- the zero vector is (0,0,...,0)
- the additive inverse $-(a_1,a_2,\ldots,a_n) = (-a_1,-a_2,\ldots,-a_n)$
- notation conventions sometimes convenient
  - $u = (u_1,u_2,\ldots,u_n)$ etc
  - lists of u vectors can be written with superscripts u^1, u^2 etc if necessary

Example: Polynomial spaces

- general polynomial space is P(t) = the set of all polynomials $p(t) = a_0 + a_1t + a_2t^2 + \ldots a_kt^k$ with coefficients $a_i \in F$, any degree k
- vector space operations:
  - p(t) + q(t) is the polynomial defined by adding all the terms in both p(t) and q(t)
  - kp(t) is the polynomial defined by multplying each term of p(t) by k
- zero vector is the polynomial with no terms at all
- the additive inverse of p(t) is the polynomial with all the terms of p(t) given opposite sign

Example: A binary vector space

- V is the collection of all subsets of a given set
- sum of two subsets E_1 and E_2 is the symmetric difference:
  $E_1+E_2 = (E_1 \cup E_2) - (E_1 \cap E_2)$
- scalar multiples (there are only two scalars) of E are defined by
  1E = E and 0E = Ø
- zero vector is Ø
- -E = E, i.e. its own additive inverse
  $E+(-E) = E + E = (E \cup E) - (E \cap E) = E - E = \varnothing$

Example: Matrices

- $M_{m,n}$ = the space of all mxn matrices (arrays of scalar entries, with m rows and n columns)

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- this matrix is often written in simple notation as $A = (a_{ij})$
- matrix sum and scalar multiple are defined component-wise:
  $A+B = (a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij})$
  $kA = k(a_{ij}) = (ka_{ij})$
- zero matrix 0 = (0)
- additive inverse of (a_{ij}) is $(-a_{ij})$

Example: Function spaces

- X = any set, V = {f:X→F}, i.e. set of all scalar-valued functions on X
- vector space operations:
  - f + g is the function defined by adding point-wise: (f+g)(x) = f(x) + g(x)
  - kf is the function defined by scalar multiplying pointwise: (kf)(x) = kf(x)
- zero vector is the function which is identically zero: f(x) = 0 all x
- the additive inverse of f(x) is the function -f defined by defined by (-f)(x) = -(f(x))
- X is typically an open interval (a,b), a closed interval [a,b], or an infinite interval $(-\infty,\infty)$

Some simple results

a) the sum $u_1 + u_2 + \ldots + u_m$ is unambiguous without parentheses
b) the zero vector is unique
c) the additive inverse -u of u is unique
d) if u+w = v+w then u = v [cancellation law]
e) subtraction of vectors can be defined by
   u-v = u+(-v)

Some more simple results

f) k**0** = **0**

$k\mathbf{0} = k(\mathbf{0}+\mathbf{0}) = k\mathbf{0} + k\mathbf{0}$
$\mathbf{0} = k\mathbf{0}+(-k\mathbf{0}) = (k\mathbf{0} + k\mathbf{0}) + (-k\mathbf{0})$
$= k\mathbf{0} + (k\mathbf{0} + (-k\mathbf{0})) = k\mathbf{0} + \mathbf{0} = k\mathbf{0}$

g) 0u = **0**

$0u = (0+0)u = 0u + 0u$
$\mathbf{0} = 0u + (-0u) = (0u + 0u) + (-0u)$
$= 0u + (0u + -(0u)) = 0u + \mathbf{0} = 0u$

And some more simple results

h) ku = **0** implies k = 0 or u = **0**

ku = **0** and $k \neq 0$ then
$u = 1u = (k^{-1}k)u = k^{-1}(ku) = k^{-1}(\mathbf{0}) = \mathbf{0}$

i) (-k)u = -ku = k(-u)

$\mathbf{0} = k\mathbf{0} = k(-u+u) = k(-u) + ku$
so k(-u) = -ku [why?]
Also $\mathbf{0} = 0u = (k - k)u = ku +(-k)u$
so (-k)u = -ku [why?]

Non-examples: NOT a vector space

V = {(a,b)$\in R^2$ so that ....}

i. (a,b) + (c,d) = (a+c,b+d) & k(a,b) = (ka,b)
   [0(a,b) = (0,b) ≠ 0 in general]
ii. (a,b) + (c,d) = (a,b) & k(a,b) = (ka,kb)
   [(a,b)+(c,d) = (a,b) ≠ (c,d) = (c,d)+ (a,b) in general]
iii. (a,b) + (c,d) = (a+c,b+d) & k(a,b) = (k^2a,k^2b)
   [$(r+s)(a,b) = ((r+s)^2a,(r+s)^2b) \neq (r^2a, r^2b) + (s^2a, s^2b) = r(a,b)+s(a,b)$ in general]
iv. (a,b)+(c,d) = (a+c,b+d) & k(a,b) = (ka,0)
   [**ALL axioms ok EXCEPT the weird one #10:**
   1(a,b) = (1a,0) = (a,0) ≠ (a,b) in general]

Subspaces

- a subset W⊂V is a *subspace* of V if it is a vector space with the operations inherited from V
- handy notation W<V (not in text)
- to confirm W is a subspace you only need to check that:
  - 0 ∈ W [or just non-empty]
  - u,v ∈ W implies u+v ∈ W [closed under vector sums]
  - u ∈ W, k ∈ F implies ku ∈ W [closed under scalar multiples]

  all the other axioms are automatic by virtue of being inherited from V
- {0} and V are ss of any vs V
- I use ‘ss’ for subspace and ‘vs’ for vector space

Examples: Euclidean subspaces

- subspaces of Euclidean space R^3 [any ss must include the origin]:
  - the origin {0}
  - a line through the origin
  - a plane through to origin
  - all of R^3
- subspaces of Euclidean space R^n
  - the origin {0}
  - ....
  - *hyperplane* $W = \{w \in R^n | \sum a_i w_i = 0\}$
  - all of R^n

How to check if W is a subspace

- in unit V [when we've learned about inner products and orthogonality] we'll learn simple ways to derive and manipulate equations of lines and (hyper)planes in R^3 (and R^n)
- for now we can check that these are subspaces
- to re-iterate we need to check two things to show that a non-empty W < V:
  1. **If $u,v \in W$ then so is u+v.**
  2. **If $u \in W$ then so is ku for any scalar k.**
- this is done by ...
  - examining the rule which defines the vectors in W and
  - checking if it is satisfied in 1 & 2 above

Check: Hyperplane is a subspace of R^n

If $u,v \in H$ then $\Sigma a_i u_i = \Sigma a_i v_i = 0$.

1. We need to check $u+v \in H$? See if u+v satisfies the equation for H....
2. We also need to check $ku \in H$? See if ku satisfies the H equation....
3. We should also check that H has at least one vector, e.g. the zero vector.

Conclusion: $H<R^n$

- the RHS in the H equation has to be zero or neither closure check works
- only hyperplanes passing through the origin are ss of R^n

Matrix transpose A^T

- the *transpose* of an mxn matrix $A = (a_{ij})$ is $A^T = (a_{ji})$
- example

$$A = \begin{bmatrix} -1 & 3 & -4 & 2 \\ 2 & 0 & -1 & -2 \end{bmatrix} \quad A^T = \begin{bmatrix} -1 & 2 \\ 3 & 0 \\ -4 & -1 \\ 2 & -2 \end{bmatrix}$$

- A is *symmetric* if $A = A^T$ and *anti-symmetric* $A = -A^T$
- these types of matrices must be square, e.g.

$$A = \begin{bmatrix} -1 & 3 & -4 \\ 3 & 0 & -1 \\ -4 & -1 & 6 \end{bmatrix} = A^T \qquad A = \begin{bmatrix} 0 & 3 & -4 \\ -3 & 0 & -1 \\ 4 & 1 & 0 \end{bmatrix} = -A^T$$

Examples: subspaces of $M_{m,n}$

- the symmetric matrices form a subspace of $M_{n,n}$
  - A,B symmetric then $(A+B)^T = A^T + B^T = A + B$ so A+B is symmetric
  - A symmetric then $(kA)^T = (ka_{ij})^T = (ka_{ji}) = kA$ so kA is symmetric
- similarly the anti-symmetric matrices form a subspace of $M_{n,n}$
- another ss of $M_{m,n}$: mxn matrices for which $a_{11} = 0$

Examples: polynomial subspaces

- P_n = the set of all polynomials of degree n or less (i.e. n is the maximum degree):
  $p(t) = a_0 + a_1 t + a_2 t^2 + \ldots a_n t^n$ (fixed n)
- $P_n(t) < P(t)$
- we have a chain of polynomial subspaces:
  $P_0(t) < P_1(t) < P_2(t) < \ldots < P_n(t) < \ldots P(t)$
- each ss is a ss of any ss higher in the chain
- note that P_0 is 'indistinguishable' from R
- another ss of P(t): all polynomials with even degree

Examples: function subspaces

- the vs of real-valued functions X to R is written *F*(X,R)
- the sets of *even* functions f so that f(-x) = f(x) and the *odd* functions f so that f(-x) = -f(x) are ss
- the set of *bounded* functions f so that |f(x)| ≤ M (some real number M) is a subspace of *F*(X,R)
- other important subspaces of *F*(X,R):
  - the *continuous* functions C^0(X,R)
  - functions with continuous first derivative C^1(X,R)
  - ...
  - functions with continuous mth derivative C^m(X,R)
  - infinitely differentiable functions C^∞(X,R)
- we have a chain of real-valued real function ss:

$$P_n < C^\infty < \cdots < C^m < \cdots < C^1 < C^0 < F(R,R)$$

Non-examples: Euclidean space

Subsets of R^3 which are NOT subspaces:

- W = {$v \in R^3$ | $v_1 \geq 0$}
- W = {$v \in R^3$ | $v_1 + v_2 + v_3 = 1$}
  [a plane not through the origin]
- W = {$v \in R^3$ | v_i is rational}

Non-examples: function space

Subsets of *F*(R,R) which are NOT subspaces:

- W = {f : f(7) = 2 + f(1)}
- W = {f : f(x) ≥ 0}

REMEDIAL DIGRESSION: complex numbers

- the set of complex numbers is C:
  {z = x + iy, x,y ∈ R} with $i^2 = -1$
- x = *Re*(z) is the *real part* and y = *Im*(z) is the *imaginary part* of z
- two concepts:
  - the complex *conjugate* of z is $\bar{z} = x - iy$
  - the *modulus* of z is |z| = √(x^2+y^2)
- ....related by $z\bar{z} = (x+iy)(x-iy) = x^2 + y^2 = |z|^2$
- if these are hazy study the examples in sec 1.7 carefully [review the stuff anyway]

Complex Euclidean space C^n

- the set of n-tuples with complex entries is a vector space C^n over R
- C^n is also a vs over C
- what's the difference?
- R^n is a subspace of C^n considered to be a vs over R, but....
- ...R^n is NOT a subspace of C^n over C, because ku can have complex entries with k ∈ C & u ∈ R^n, so ku would not be in R^n in general

Example: subspaces of the binary vs

- let V be the binary vector space defined on the collection of subsets of the set {1,2,3,4}
- let W = {∅,123, 124, 34}, where 123 is a short notation meaning the set {1,2,3} etc.
- is W a ss of V?
- it's trivially closed under scalar multiples
- check that W is closed under vector sums:

  123 + 124 = 34 ∈ W
  123 + 34 = 124 ∈ W
  124 + 34 = 123 ∈ W
  123 + 124 + 34 = ∅ ∈ W

Motivational address

- is R a subspace of R^3?
- geometrically the x-axis is a line in Euclidean space, or...
- ...it is a line (i.e. R) without reference to its being part of Euclidean space
- in this obvious sense R *is* a subspace of R^3
- R can be considered the 'same' as the true ss:
  $W = \{(x,y,z)\in R^3 : y = z = 0\}$
  i.e. all points in R^3 of the form (a,0,0), $a\in R$
- this is a subtle but important distinction

More motivational address

- we will develop concepts that make precise the meaning of 'sameness' for vs ... **isomorphism**
- in this sense any '3-dimensional' vs will be the 'same' as R^3
- this is why all the vs examples we've looked at seem to be variations on a theme (except binary)
- we have no precise meaning for the concept of **dimension** yet either, only intuition
- also, if all n-dimensional vs are really the 'same' why do we develop all the different examples?

... and more

- vectors in R^3 can be considered as entities v motivated by the geometric approach, or...
- ... choose a reference frame, e.g. unit vectors
  i = (1,0,0) j = (0,1,0) k = (0,0,1)

  and express any vector $v\in R^3$ in terms of these:

  $v = v_1 i + v_2 j + v_3 k = (v_1, v_2, v_3)$
- these are the coordinates of v with respect to the standard unit vectors
- if we choose different unit vectors we get different coordinates for the same vector v [e.g. frame of reference in physics], but....

...and even more

- ...no matter how we do this we'll always need exactly the same number of unit vectors
- this number of coordinates required defines the dimension of R^3
- the same concept can be used to define the dimension of a general vs .... a **basis**
- in the R^3 example we had to be careful to choose a special set of vectors to define the coordinate axes
- for instance (1,0,0), (0,1,0) & (1,1,0) wouldn't work because (1,1,0) = (1,0,0)+(0,1,0)
- the concept of **linear independence** handles this problem in the case of a general vs

We need some more general vs concepts

- sameness, dimension, linear independence & basis
- now the other question: if all n-dimensional vs are really the 'same' why do we develop all the different examples?
- the answer is central to the problem-solving utility of linear algebra in applications:

  ...because the powerful methods familiar from R^3 can be applied to study and analyse problems involving a wide variety of different entities (i.e. different types of vectors)

Linear combinations

- V is a vs over F
- choose some vectors $\{ u_1, u_2, ...u_r \}$
- for any scalars $k_1, k_2,...,k_r$ we can evaluate

  $$w = k_1u_1 + k_2u_2 + ... + k_ru_r$$

  and it's guaranteed to be a vector in V too (why?)
- w is called a *linear combination* (lc) of $u_1, u_2, ..., u_r$

Examples: linear combinations

- in R^4 the vector (-1,1, 6,11) is a lc of the vectors (1,2,0,4) & (1,1,-2,-1) ...
- in $P_3(t)$ the polynomial $p(t) = 6 + 3t^2 - 4t^3$ is a lc of the polynomials $p_0(t) = 1$, $p_1(t) = t$, $p_2(t) = t^2$, $p_3(t) = t^3$ ...

REMEDIAL DIGRESSION: solving systems

You should be able to solve the system:

$$\begin{aligned} x + 2y \quad &= 1 \\ -2x - 3y + z &= 4 \\ 5x \quad + 3z &= -3 \end{aligned}$$

Example: linear combinations

Express the vector $t^2+4t-3 \in P_2$ as a lc of the vectors:
$\{t^2-2t+5, 2t^2-3t, t+3\}$ ..... [two solution approaches]

Linear span

- given a set of vectors $\{u_1, u_2, ...u_r\}$ in a vs V
- we can form the set W of all vectors which are lc's of these u_i vectors
  - $W = \{w \in V : w = k_1u_1 + k_2u_2 + ... k_ru_r\}$
  - k_i any scalars
- W is called the *(linear) span* of the set of vectors $S = \{u_1, u_2, ...u_r\}$, or ...
- W is the (vector) space *spanned by* S
- S is called a *spanning set* for W
- we write $W = Sp\{u_1, u_2, ...u_r\}$ or $W = Sp(S)$

Linear span

- Sp(S) is the smallest subspace of V containing the vectors $u_1, u_2, ...u_r$
  - it's definitely a subspace because it's a subset and closed to all vector sums and scalar products of the $u_1, u_2, ...u_r$
  - as well .... *any* ss W of V containing the S vectors must also contain all lc of them, i.e. $Sp(S) \subset W$
  - in fact the two closure checks for a ss are equivalent to being closed under linear combinations of its vectors

Example: Linear span

Show that $Sp\{(1,1,1),(1,2,3),(1,5,8)\} = R^3$ [4.13 text]

- to show this requires that any vector $(a,b,c) \in R^3$ can be written as a lc of these three vectors:

  $(a,b,c) = x(1,1,1) + y(1,2,3) + z(1,5,8)$
- we have a system of equations

  $$\begin{aligned} x+y+z &= a \\ x+2y+5z &= b \\ x+3y+8z &= c \end{aligned} \quad \text{which reduces to} \quad \begin{aligned} x+y+z &= a \\ y+4z &= b-a \\ z &= -c+2b-a \end{aligned}$$
- the (unique) solution for the lc is

  $x = -a+5b-3c$, $y = 3a-7b+4c$, $z = -a+2b-c$
- for example

  $(1,6,-2) = 35(1,1,1) - 47(1,2,3) + 13(1,5,8)$

  $(0,-1,-2)= 1(1,1,1) - 1(1,2,3) + 0(1,5,8)$ etc

Example: Linear span

Find W = Sp{(1,1,1),(1,2,3),(3,5,7)}

- a vector (a,b,c)∈W means it can be written as a lc of the three spanning vectors given:

  (a,b,c) = x(1,1,1) + y(1,2,3) + z(3,5,7)
- we have a system of equations

x+y+3z = a	which	x+y+3z = a
x+2y+5z = b	reduces to	y+2z= b-a
x+3y+7z = c		0 = a-2b+c

- there is no solution unless a-2b+c=0 in which case

  z = k (arbitrary), y = b-a-2k, x = 2a-b-k
- the condition a-2b+c=0 implies only vectors of the form (a,b,2b-a) = a(1,0,-1)+b(0,1,2) are in W
- this shows that W is also spanned by just two vectors:

  W = Sp{(1,0,-1),(0,1,2)}

Linear independence

- any vector w ∈ Sp(u_1, u_2, ..., u_r) can be expressed as a lc w = $k_1u_1 + k_2u_2 + ... + k_ru_r$ for some scalars k_i
- the zero vector can certainly always be expressed this way with all of the k_i = 0, but...
- ... if there is some non-zero lc which gives the zero vector then the set {u_1, u_2, ...u_r} is called *linearly dependent*
- equivalently {u_1, u_2, ...u_r} is a *linearly independent* set of vectors if $k_1u_1 + k_2u_2 + ... + k_ru_r = 0$ implies all the k_i = 0

Simple examples: Linear independence

- {0} is always linearly dependent [why?]
- any set which includes the zero vector is linearly dependent
- any set {v} with one single vector v ≠ 0 is linearly independent [why?]
- a set of two non-zero vectors {u,v} is linearly dependent if and only if u = kv, i.e. one is a scalar multiple of the other
- with e_i = (0,...,0,1,0,....,0), i.e. zero everywhere except for a 1 in the ith position, the set {e_1, e_2, ..., e_n} is linearly independent in R^n

Simple examples: Linear independence

- the set {1, t, t^2, ..., t^n} is linearly independent in P_n
- in C(-∞,∞) the set {1, x, cos x} is linearly independent
- in C(-∞,∞) the set {1, x, $\cos^2$x, $\sin^2$x} is linearly dependent
  - because 1 - $\cos^2$x - $\sin^2$x = 0 (identically zero) is a non-zero lc of the vectors that adds to the zero function
- in the binary vs on {1,2,3,4} the set {123,124, 34} is linearly dependent
  - 123 + 124 + 34 = Ø is a non-zero lc of vectors giving the zero vector

Connection between linear dependence and span

- a set {u_1, u_2, ...,u_r} is linearly dependent if and only if at least one u_i is in the span of the other r-1 vectors
  - dependency implies there is a lc $k_1u_1 + k_2u_2 + ... + k_ru_r = 0$ and at least one $k_i \neq 0$
  - so we can solve for $u_i = -(k_1/k_i)u_1 - (k_2/k_i)u_2 - ... - (k_r/k_i)u_r$ (with the ith rhs term omitted)
  - this shows that u_i is a non-zero lc of the other r-1 vectors, i.e. u_i ∈ Sp{u_1, u_2, ..., u_{i-1}, u_{i+1}, ... , u_r}
  - to prove the converse suppose that one vector is in the Sp of the other r-1 vectors (re-number the list so it's u_1)
  - we can write say $u_1 = c_2u_2 + c_3u_3 + ... + c_ru_r$ with not all c_i = 0
  - re-arranging gives a non-zero lc $u_1 - c_2u_2 - c_3u_3 - ... - c_ru_r = 0$ so the vectors are linearly dependent

Linear dependence depends on the scalars

- consider the vectors u=(1+i,2i) & v=(1,1+i) ∈C^2
- u&v are linearly dependent over complex scalars:
  - assume u = kv and solve for k
  - second coordinate: 2i = k(1+i) so k = 2i/(1+i) = 1+i
  - this k also satisfies the first coordinate in u = kv since 1+i = k1 = 1+i
- BUT u&v are linearly independent over the real scalars:
  - ...because u = kv implies k = 1+i ∈C so u is not a real scalar multiple of v

Checking linear independence

- are the vectors {u,v,w} where u=(1,-2,1), v=(2,1,-1), w=(7,-4,1) linearly dependent or independent?
- set (0,0,0) = xu + yv + zw and solve for x,y,z
- this is
$$\begin{aligned} x+2y+7z &= 0 \\ -2x+y-4z &= 0 \\ x-y+z &= 0 \end{aligned}$$
- before reducing the system let's streamline the work by using a matrix of coefficients $\begin{bmatrix} 1 & 2 & 7 \\ -2 & 1 & -4 \\ 1 & -1 & 1 \end{bmatrix}$
- note this is just the matrix with the [u | v | w] as columns

Checking linear independence

- so to check linear independence you can write the given vectors as columns of a matrix and *row-reduce* it:
$$\begin{bmatrix} 1 & 2 & 7 \\ -2 & 1 & -4 \\ 1 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 7 \\ 0 & 5 & 10 \\ 0 & -3 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$
- this procedure is like reducing a system of equations
- in <u>this</u> example there is a non-zero solution (in fact an infinite number of them) and so the vectors are linearly dependent
- if we had got only the zero solution we would conclude that the vectors are linearly independent because only the zero lc gives the zero vector

Row and column space of a matrix

- let A be an m×n matrix
- the rows of A can be considered as vectors R_1, R_2, ...,$R_m \in R^n$
- the subspace of R^n spanned by the rows of A is called the *row space* of A
- the columns of A can be considered as vectors C_1, C_2, ...,$C_n \in R^m$
- the subspace of R^m spanned by the columns of A is called the *column space* of A

Row equivalent matrices

- suppose B is obtained from A by a sequence of the following operations:
  1. interchance two rows R_i and R_j
  2. replace a row R_i by a scalar multiple kR_i
  3. replace a row R_i by $R_i + R_j$
- these are called *elementary row operations* on A
- we write B ~ A
- B is said to be *row equivalent* to A

Row equivalent matrices

- elementary row operations can be viewed as operations on vectors in R^n (rows of A):
  - interchange vectors R_i and R_j
  - multiply a vector R_i by a scalar k
  - replace a vector R_i by the sum $R_i + R_j$
- these operations do not affect the space Sp{R_1, R_2, ..., R_n} spanned by the rows
  - the order of the spanning vectors is irrelevant
  - vectors are replaced by linear combinations with other vectors
  - no vectors are eliminated
- **row equivalent matrices have <u>identical</u> row spaces**

Using row operations to check linear dependence

- this result should be considered along with the technique described on slides 49&50
- we can check linear independence by
  - arranging the vectors as the <u>rows</u> of a matrix
  - performing row operations until the matrix is in echelon form
  - if there are less rows than the number of vectors then the original vectors are linearly dependent
- two questions:
  - what is echelon form?
  - why did we use columns before [slide 49]?

Echelon matrix

- we'll defer the answer to the second question
- the leading non-zero entry in a row is called the *pivot entry* of the row
- an *echelon matrix* is in the following form:
  - all zero rows are at the bottom of the matrix
  - the pivot entry in a row is in a column to the right of the pivot entry in the preceding row
- **the rows of an echelon matrix are linearly independent** [why?]

Basis of a vector space

- let V = Sp{u_1, u_2, ...,u_r}
- if w is any vector in V then {w, u_1, u_2, ...u_r} is certainly linearly dependent
- the spanning set itself {u_1, u_2, ...u_r} may or may not be linearly dependent, but if it's linearly dependent....
- ...choose a u_i which is a lc of the other r-1 vectors [slide 47]
- then V is also spanned by just those r-1 vectors {u_1,..., u_{i-1}, u_{i+1},...,u_r}
- continuing to *'cast out'* dependent vectors, we eventually arrive at a linearly independent set that still spans V
- this is called a *basis* of V (the plural is *bases*)

Basis of a vector space

- a spanning set {u_1, u_2, u_3, ... , u_n} for a vs V is a *basis* if:
  - it is linearly independent, OR equivalently
  - the expression of any vector in terms of basis vectors is unique: $u = a_1u_1 + a_2u_2 + \ldots + a_nu_n$
- **all bases {u_1, ...,u_n} for V have the same number of vectors n**
  - not obvious - see proof 4.36 text
  - n is called the *dimension* of V
  - V is *n-dimensional*
  - write dim V = n
- some vector spaces may be infinite dimensional (e.g. function spaces, polynomial space P)
- think of a basis as a maximal linearly independent set

Examples: standard bases

- the *standard basis* in R^n is the set of vectors {e_1,e_2,...,e_n} defined on slide 45
  - R^n is n-dimensional
  - in R^3 these are written i = (1,0,0), j = (0,1,0), k = (0,0,1)
- in P_n the *mononomial basis* is {1, t, t^2, t^3,..., t^n}
  - dim(P_n) = n+1
- the *standard basis* for the space of m×n matrices $M_{m,n}$ consists of the set of matrices {E_{ij}} in which the ijth entry is 1 and all other entries are zero
  - dim($M_{m,n}$) = mn
- the binary vs on the set {1,2,...,n} has a basis {{1},{2},...{n}}
  - this vs is n-dimensional

Some useful results about vs bases

- if V is an n-dimensional vs then
  - any n+1 vectors in V must be linearly dependent
  - any set of n linearly independent vectors is a basis of V
  - any spanning set with n vectors is a basis of V
- examples
  - any four vectors in R^3 must be dependent
  - the vectors {1,1,1),(1,2,3),(2,-1,1)} are linearly independent [check as per method on slide 54] so they are a basis of R^3.....

Example: basis of a vector space

- ...show the set of vectors S = {(1,1,1),(1,2,3),(2,-1,1)} is a basis for R^3
- three vectors in R^3 are a basis if and only if they are linearly independent
- write as rows of a matrix and reduce to echelon form:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

- three independent rows in the echelon matrix so the vectors are independent
- S is a basis for R^3

Example: basis of a vector space

- check if the set of vectors S = {(1,1,2),(1,2,5),(5,3,4)} is a basis for R^3
- three vectors in R^3 are a basis if and only if they are linearly independent
- write as rows of a matrix and reduce to echelon form:

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 5 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & -2 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

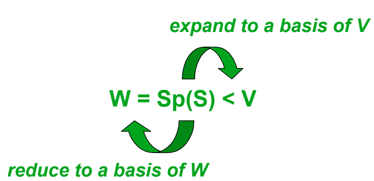
- only two independent rows in the echelon matrix so the original vectors are linearly dependent
- S is NOT a basis for R^3

Subspaces of R^n

- a subspace W < R^3 can have dimension no more than 3
- the (geometric) possibilities are:
  - point - the origin (dim W = 0)
  - line through the origin (dim W = 1)
  - plane through the origin (dim W = 2)
  - all of R^3 (dim W = 3)
- a subspace of R^n can have dimension no more than n
- the hyperplane we defined for R^n [slide 21] is a ss of dimension n-1

Bases and subspaces

- any spanning set S of a finite dimensional vs W contains a basis B obtained from S by deleting any vector that is a linear combination of the preceding vectors in S
- any set S = {u_1,...,u_r} of linearly independent vectors in a finite dimensional vs V can be extended to a basis B of V



Example: bases and subspaces

- let W = Sp{(1,-2,5,-3),(2,3,1,-4),(3,8,-3,-5)} < R^4
- find a basis for W and the dimension of W, and extend this basis to a basis for all of R^4
- write the given vectors as rows of a matrix and reduce to echelon form:

$$\begin{bmatrix} 1 & -2 & 5 & -3 \\ 2 & 3 & 1 & -4 \\ 3 & 8 & -3 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 14 & -18 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- the non-zero rows of the echelon matrix form a basis of the row space of the matrix, i.e. W
- so dim(W) = 2 with basis {(1,-2,5,-3),(0,7,-9,2)} ...

...Example: bases and subspaces

- to extend this to a basis for R^4 requires four linearly independent vectors including the two found above
- the simplest way to do this is to write an echelon matrix

$$\begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- the row vectors in this are linearly independent (as are the rows of ANY echelon matrix)
- the required basis (obviously not unique) for R^4 is therefore {(1,-2,5,-3),(0,7,-9,2),(0,0,1,0),(0,0,0,1)}

Coordinates

- let V be a finite dimensional vs over F
- choose an ordered basis B = {u_1,...,u_n}
- let u ∈ V be expressed as u = a_1u_1 + ... + a_nu_n in terms of the selected basis
- the scalars a_1,a_2,...,a_n are called the *coordinates* of u *with respect to the basis* B
- this defines an n-tuple $[u]_B = [a_1,a_2,...,a_n] \in F^n$ called the *coordinate vector* of u with respect to B
- choose a different basis B′ = {v_1,...,v_n} and express u in terms of B′: u = b_1v_1 + ... + b_nv_n
- the vector u is still the same but its coordinate vector $[u]_{B'} = [b_1,b_2,...,b_n]$ is now different

Coordinates of vectors in R^n

- the way vectors in R^n are written as n-tuples implicitly assumes a coordinate representation with respect to the standard basis S
- with respect to S a vector $v = (a_1, ..., a_n) \in R^n$ has coordinates simply $a_1, ..., a_n$, i.e. $[v]_S = [a_1, ..., a_n]$
- vectors in R^n can be represented with respect to other bases as convenient
- this is called a *change of coordinates*
- useful in dynamics for instance when we change frame of reference in R^3

Example: changing coordinates

- find the coordinates of the vector v = (2,3,4) with respect to the basis B = {(1,1,1),(1,1,0),(1,0,0)} of R^3
- write (2,3,4) = x(1,1,1) + y(1,1,0) + z(1,0,0)
- this gives the system of equations

$$\begin{aligned} x + y + z &= 2 \\ x + y &= 3 \\ x &= 4 \end{aligned}$$

- solution is x = 4, y = -1, z = -1
- so (2,3,4) = 4(1,1,1) -(1,1,0) - (1,0,0)
- $[v]_B = [4,-1,-1]_B$ are the new coordinates of (2,3,4) with respect to B
- this simple example should illustrate the concepts

Example: coordinates in polynomial space

- find a basis for $W = Sp\{v_1,v_2,v_3,v_4\} < P_3$ where
  - $v_1 = t^3 - 2t^2 + 4t + 1$
  - $v_2 = 2t^3 - 3t^2 + 9t + 1$
  - $v_3 = t^3 + 6t - 5$
  - $v_4 = 2t^3 - 5t^2 + 7t + 5$
- the coordinates of these polynomials with respect to the monomial basis $\{t^3, t^2, t, 1\}$ are
  - $[v_1] = (1, -2,4,1)$
  - $[v_2] = (2, -3,9,1)$
  - $[v_3] = (1, 0,6,-5)$
  - $[v_4] = (2,-5,7,5)$ ....

...Example: coordinates

- we can check the original polynomials by checking the coordinate vectors in R^4
- write them as rows of a matrix and reduce to echelon form

$$\begin{bmatrix} 1 & -2 & 4 & 1 \\ 2 & -3 & 9 & -1 \\ 1 & 0 & 6 & -5 \\ 2 & -5 & 7 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 4 & 1 \\ 0 & 1 & 1 & -3 \\ 0 & 2 & 2 & -6 \\ 0 & -1 & -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 4 & 1 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- the non-zero rows in the echelon matrix form a basis for the row space, i.e. the space of coordinate vectors, so...
- ...the corresponding vectors $\{t^3-2t^2+4t+1, t^2+t-3\}$ are a basis for W and dim(W) = 2

Isomorphism

- there is a one:one correspondence between vectors in an n-dimensional vs V over F and vectors in F^n
  - associate a vector v with its coordinates $[v]_B$ with respect to some basis $B = \{u_1,...,u_n\}$ of V
- this correspondence also preserves the vs operations
  - let $v = a_1u_1 + ... + a_nu_n$ and $w = b_1u_1 + ... + b_nu_n$
  - $[v] + [w] = [a_1,... ,a_n] + [b_1,... ,b_n] = [a_1+b_1,... ,a_n+b_n] = [v+w]$
  - $k[v] = k[a_1,... ,a_n] = [ka_1,... ,ka_n] = [kv]$
- V and F^n are called *isomorphic*, i.e. 'the same'
- we write $V \cong F^n$
- we can solve problems in other vs's by using vectors in R^n for the calculations [e.g. the previous example]

Basis finding: row space algorithm

- to find a basis for the space W spanned by a given set of vectors we can:
  - write them as the rows of a matrix R
  - W is the row space of R
  - row reduce the matrix R to echelon form
  - select the non-zero rows of the echelon matrix
  - these span the row space of R [linear combinations of rows of R] and are linearly independent [echelon form]
  - so they give the required basis
- the dimension of the row space of a matrix A is called the *rank* of A

Basis finding: casting out using column vectors

- an alternative method to find a basis for the space W spanned by a given set of vectors:
  - write them as the columns of a matrix C
  - this represents the system of equations obtained when writing an arbitrary vector as a lc of the given ones [see slides 42&43]
  - row reduce [i.e. using row operations] matrix C to echelon form
  - columns in the echelon matrix that don't have pivots correspond to arbitrary coefficients in the lc
  - the corresponding vectors in C can therefore be expressed in terms of the vectors that do match columns with pivots
  - so cast all the dependent vectors out and retain only the vectors corresponding to columns with pivots to give the basis for W
- is this consistent with the row space algorithm??

Example: basis finding (casting out algorithm)

- repeat example slide 64 using the casting out algorithm
- write the given vectors as the columns of a matrix C and row reduce to echelon form:

$$\begin{bmatrix} 1 & 2 & 3 \\ -2 & 3 & 8 \\ 5 & 1 & -3 \\ -3 & -4 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 7 & 14 \\ 0 & 9 & 18 \\ 0 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- recall this is motivated by writing an arbitrary vector (a,b,c,d)$\in R^4$ as:

  (a,b,c,d) = x(1,-2,5,-3)+y(2,3,1,-4)+z(3,8,-3,-5)

  and solving for the unknown coefficients in the lc...

...Example: basis finding (casting out algorithm)

- any vector v$\in$W must satisfy the restrictions on a,b,c,d that come from the zero rows in the echelon matrix
- the non-zero rows in the echelon matrix indicate that z is arbitrary in the lc that expresses v
- so only the first two vectors are necessary in this lc
- we conclude that these are a basis for W:

  {(1,-2,5,-3),(2,3,1,-4)}
- in particular W is two dimensional as found previously
- in this example:
  - rank of R = 2
  - rank of C = 2
- there is no discrepancy between the methods - they each find a valid basis for W

Casting out algorithm

- another way of thinking of the casting out algorithm...
- suppose A ~ M echelon form
- the columns in M with pivot entries are a basis for the column space M
- the corresponding columns in A are a basis for the column space of A [why?]
- examine a simple example to see why this works
- more on this result when we study solution spaces of linear systems...

An **IMPORTANT** result about rank

- for any matrix A: **rank(A) = rank(A^T)**
  - when we use the row space algorithm vs the casting out algorithm the matrices are transposes [$C=R^T$]
  - so we'll always get the same number of independent rows in either of the reduced echelon forms
  - i.e. the found bases will have the same number of vectors....as they should
- we can also conclude for any matrix A that **the row and column space have the same dimension**
- proving these is more involved than difficult - see text

Example: rank of a matrix

- find the ranks of the following matrices:

$$A = \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & 0 \\ -2 & -1 & 3 \\ -1 & 4 & -2 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 3 \\ 0 & -2 \\ 5 & -1 \\ -2 & 3 \end{bmatrix}$$

$$A \sim \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & -3 & -6 & -3 & 3 \\ 0 & -1 & -2 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- an echelon matrix with two non-zero rows
- so rank(A) = 2 ....

....Example: rank of a matrix

- row reduce B^T instead of B [less work involved]:

$$B^T \sim \begin{bmatrix} 1 & 2 & -2 & -1 \\ 0 & -3 & 3 & 6 \\ 0 & 6 & -3 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -2 & -1 \\ 0 & -3 & 3 & 6 \\ 0 & 0 & 3 & 7 \end{bmatrix}$$

- an echelon matrix with three non-zero rows
- so rank(B) = rank(B^T) = 3
- as for the last matrix....rank(C^T) = 2 since the rows of C^T are linearly independent [i.e. not multiples of each other]
- therefore rank(C) = rank(C^T) = 2
- use properties of rank to simplify work as in this example

Example: basis finding again

Find bases for the row space and column space of A

$$A = \begin{bmatrix} 1 & 1 & 3 & 3 \\ 0 & 2 & 2 & 4 \\ 1 & 0 & 2 & 1 \\ 1 & 1 & 3 & 3 \end{bmatrix}$$

- first reduce A to echelon form:

$$A \sim \begin{bmatrix} 1 & 1 & 3 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

- the two non-zero rows of R are a basis for the row space of A: {(1,1,3,3), (0,1,1,2)}
- the two columns of R with pivot entries are a basis for the column space of R so .....
- .... the corresponding columns of A are a basis for the column space of A: $\{(1,0,1,1)^T,(1,2,0,1)^T\}$

....Example: basis finding again

Now find bases for the row (column) space of A consisting only of rows (columns) of A

- previous answer is ok for the column space
- for the row space we need to be clever .....
- use A^T and the casting out algorithm:

$$A^T = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 2 & 0 & 1 \\ 3 & 2 & 2 & 3 \\ 3 & 4 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R'$$

- columns 1&2 of A^T are a basis for the column space of A^T and so ....
- ...rows 1&2 of A are a basis for the row space of A: {(1,1,3,3),(0,2,2,4)}
- compare this answer to the one on the previous slide

A question for understanding

- two row equivalent matrices A~B are related by a sequence of row operations
- so the rows of B are l.c. of the rows of A, consequently...
- A and B have the same row space
- what about column space?
- do row equivalent matrices A and B have the same column space?

ANSWER

Sums of subspaces

- let V be a v.s. and take two subsets U,W⊂V
- the *sum* U+W = {u+w|u∈U, w∈W} consists of all v.s. sums of vectors in the two subsets
- now if U,W are subspaces of V then U+W is also a subspace [closure is easily checked]
- in fact U+W = span{U,W}
- also the intersection of two subspaces U∩W is a subspace of V
- if U,W are finite dimensional subspaces of V then

  dim(U+W) = dim U + dim W - dim(U∩W)
- this result is very IMPORTANT

Example: Sums of subspaces

- U = {(a,b,0)∈R^3} is the xy-plane and W = {(0,c,d)∈R^3} is the yz-plane
- the sum U+W = R^3
- the intersection U∩W = {(0,c,0)∈R^3} is the y-axis
- dim(U+W) = dim U + dim W - dim(U∩W) = 2 + 2 - 1 = 3 as it should
- a vector (a,b,c) ∈R^3 can be written as a sum of vectors in U and W, but not uniquely, e.g.

  (2,16,-12) = (2,4,0) + (0,12,-12)

  = (2,-8,0) + (0,24,-12) etc

Direct sums

- V is the *direct sum* of the subspaces U and W if
  V= U+W <u>AND</u> U∩W = {0}
- we write V = U⊕W for the direct sum
- the importance of direct sum:
  V = U⊕W if and only if any vector v∈V can be written <u>uniquely</u> as a sum v=u+w, u∈U, w∈W
- if V = U⊕W then dim(U∩W) = 0 (second condition above) so in this case:
  dim(U⊕W) = dim U + dim W

Example: Direct sums of subspaces

- U = {(a,b,0)∈R^3} is the xy-plane and W = {(0,0,d)∈R^3} is the z-axis
- then the <u>direct</u> sum U⊕W = R^3
- dim(U+W) = dim U + dim W = 2 + 1 = 3 as it should
- a vector (a,b,c) ∈R^3 can be written <u>uniquely</u> as a sum of vectors in U and W, e.g.
  (2,16,-12) = (2,16,0) + (0,0,-12)
  and <u>no other sum</u> of these kinds of vectors will work

Example: sums of subspaces

- compare this to text problem 4.54, but note typo in a) conclusion
- U = Sp{(1,4,0,-1),(2,-3,1,1)} and W = Sp{(0,1,1,1),(4,5,1,-1)}
- find bases for U+W and U∩W
- U+W is the span of the four vectors, so write as rows and find a basis for the row space:

$$\begin{bmatrix} 1 & 4 & 0 & -1 \\ 2 & -3 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 4 & 5 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 0 & -1 \\ 0 & 11 & -1 & -3 \\ 0 & 1 & 1 & 1 \\ 0 & 11 & -1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 0 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 6 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- the first three rows of the echelon matrix {(1,4,0,-1),(0,1,1,1),(0,0,6,7)} are a basis for U+W
- now, to find U∩W, characterize vectors in both U and W
- assume (x,y,z,w) ∈ U first, so (x,y,z,w) = a(1,4,0,-1)+b(2,-3,1,1)...

...Example: sums of subspaces

- then

$$\begin{bmatrix} 1 & 2 & x \\ 4 & -3 & y \\ 0 & 1 & z \\ -1 & 1 & w \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & x \\ 0 & 11 & 4x-y \\ 0 & 1 & z \\ 0 & 3 & x+w \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & x \\ 0 & 1 & z \\ 0 & 0 & 11z-4x+y \\ 0 & 0 & 3z-x-w \end{bmatrix}$$

- so we have consistency conditions on x,y,z,w as per the last two rows
- now do the same with (x,y,z,w)∈W so that
  (x,y,z,w) = c(0,1,1, 1)+d(4,5,1,-1) and reduce to echelon form

$$\begin{bmatrix} 0 & 4 & x \\ 1 & 5 & y \\ 1 & 1 & z \\ 1 & -1 & w \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & z \\ 0 & 4 & x \\ 0 & 4 & y-z \\ 0 & 6 & y-w \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & z \\ 0 & 4 & x \\ 0 & 0 & x-y+z \\ 0 & 0 & 3x-2y+2w \end{bmatrix}$$

...Example: sums of subspaces

- again consistency for the last two rows provides conditions on the vector if (x,y,z,w)∈W
- arranging both sets of consistency conditions into one system:

$$\begin{aligned} x-y+z &= 0 \\ 3x-2y+2w &= 0 \\ -4x+y+11z &= 0 \\ -x+3z-w &= 0 \end{aligned}$$

- solve this by reducing the matrix of coefficients

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 3 & -2 & 0 & 2 \\ -4 & 1 & 11 & 0 \\ -1 & 0 & 3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -3 & 2 \\ 0 & 1 & -5 & 0 \\ 0 & -1 & 4 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- w is arbitrary, so pick say w = -1, then z = 1, y = 5, x = 4
- the required basis for U∩W is then {(4,5,1,-1)}

Vector spaces: Roadmap

- definition of a vector space
  - axioms
  - elementary results
  - standard examples
    - Euclidean space R^n and C^n [NB review of complex arithmetic]
    - matrix space
    - function spaces
    - polynomial spaces
    - binary vector space
- subspaces
  - definition
  - standard examples
  - checking by closure

Vector spaces: Roadmap

- linear combinations and span
- matrix spaces
  - row space and column space of a matrix
  - elementary row operations
  - row equivalent matrices
  - echelon form
- linear independence
  - basic results
  - how to check it for a set of vectors

Vector spaces: Roadmap

- basis of a vector space
  - definition
  - dimension (finite dimensional v.s.)
  - basis finding methods
    - [solving the linear combination with arbitrary constants]
    - rowspace method
    - column method (casting out)
- rank of a matrix
  - rank A = rank A^T
- vector space sums
  - $\dim(U+W) = \dim U + \dim W - \dim(U \cap W)$
  - direct sum $U \oplus W$ provides unique decomposition