Linear Algebra, Spring 2005

Solutions

May 4, 2005

Problem 6.47 (d)

Since E is the usual basis, the change-of-basis matrix P from E to S, can be written directly from S by writing the basis vectors in S as its columns.

$$P = \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}$$

Two methods are given to find the change-of-basis matrix Q from S back to E

Method 1. Directly from P, as $Q = P^{-1}$, find P^{-1} using matrix inversion formula:

$$Q = P^{-1} = \begin{bmatrix} -\frac{5}{2} & 2 \\ \frac{3}{2} & -1 \end{bmatrix}$$

Method 2. Use the definition of the change-of-basis matrix. i.e. express each vector in E as a linear combination of the vectors in S. Let $v = (a, b)$ be any arbitrary vector in E, then calculate the coordinates of v with respect to S as:

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} x + \begin{bmatrix} 4 \\ 5 \end{bmatrix} y \Longrightarrow \begin{cases} 2x + 4y = a \\ 3x + 5y = b \end{cases} \Longrightarrow \begin{cases} x = -\frac{5}{2}a + 2b \\ y = \frac{3}{2}a - b \end{cases}$$

Hence, $[v]_S = \begin{bmatrix} -\frac{5}{2}a + 2b \\ \frac{3}{2}a - b \end{bmatrix}$

Use above formula to get the coordinates of e_i

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} -\frac{5}{2} \\ \frac{3}{2} \end{bmatrix}; \qquad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Write the coordinates of e_i as columns of change-of-basis matrix Q

$$Q = \begin{bmatrix} -\frac{5}{2} & 2 \\ \frac{3}{2} & -1 \end{bmatrix}$$

Problem 6.48

(a) Transition matrix P

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \end{bmatrix} + y \begin{bmatrix} 2 \\ 3 \end{bmatrix} \Longrightarrow \begin{cases} x + 2y = 1 \\ 2x + 3y = 3 \end{cases} \Longrightarrow \begin{cases} x = 3 \\ y = -1 \end{cases}$$

$$\begin{bmatrix} 1 \\ 4 \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \end{bmatrix} + y \begin{bmatrix} 2 \\ 3 \end{bmatrix} \Longrightarrow \begin{cases} x + 2y = 1 \\ 2x + 3y = 4 \end{cases} \Longrightarrow \begin{cases} x = 5 \\ y = -2 \end{cases}$$

Hence, $P = \begin{bmatrix} 3 & 5 \\ -1 & -2 \end{bmatrix}$

(b) Transition matrix Q

$$Q = P^{-1} = \begin{bmatrix} 2 & 5 \\ -1 & -3 \end{bmatrix}$$

It can also be obtained using method as in part a.

Problem 6.49

Part (a)

Using simple geometry, find the coordinates of the new unit vectors when the original axis is rotated by $30°$ counterclockwise

Coordinates of ${e_i}'$:

${e_1}' = (\cos 30, \sin 30) = (\frac{\sqrt{3}}{2}, \frac{1}{2})$

${e_2}' = (-\cos 60, \sin 60) = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$

Part (b)

The change-of-matrix P can simply be written as the columns of the two unit vectors e_i'

$$P = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

Part (c)

Any point $X = [x, y]$ will have new coordinates $X' = P^T X^T$

$$A' = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2.37 \\ 2.1 \end{bmatrix}$$

$$B' = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 2 \\ -5 \end{bmatrix} = \begin{bmatrix} -0.77 \\ -5.33 \end{bmatrix}$$

$$C' = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2}a + \frac{1}{2}b \\ -\frac{1}{2}a + \frac{\sqrt{3}}{2}b \end{bmatrix}$$

Problem 6.50

Part (a)

Since E is the usual basis, the change-of-basis matrix P from E to S, can be written directly from S by writing the basis vectors in S as its columns.

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

Use the definition of the change-of-basis matrix. i.e. express each vector in E as a linear combination of the vectors in S. Let $v = (a, b, c)$ be any arbitrary vector in E, then calculate the coordinates of v with respect to S as:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} y + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} z \Longrightarrow \begin{cases} x = a \\ y = a - b + c \\ z = -2a + 2b - c \end{cases}$$

$$[v]_S = \begin{bmatrix} a \\ a - b + c \\ -2a + 2b - c \end{bmatrix}$$

Coordinates of e_i:

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}; \; e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}; \; e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Write the coordinates of e_i as columns of Q

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ -2 & 2 & -1 \end{bmatrix}$$

Q can also be obtained by taking the inverse of P. i.e. $Q = P^{-1}$

Part (b)

Since E is the usual basis, the change-of-basis matrix P from E to S, can be written directly from S by writing the basis vectors in S as its columns.

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix}$$

Use the definition of the change-of-basis matrix. i.e. express each vector in E as a linear combination of the vectors in S. Let $v = (a, b, c)$ be any arbitrary vector in E, then calculate the coordinates of v with respect to S as:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} y + \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} z \Longrightarrow \begin{cases} x = -2b + c \\ y = 2a + 3b - 2c \\ z = -a - b + c \end{cases}$$

$$[v]_S = \begin{bmatrix} -2b + c \\ 2a + 3b - 2c \\ -a - b + c \end{bmatrix}$$

Coordinates of e_i:

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}; \ e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix}; \ e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Write the coordinates of e_i as columns of Q

$$Q = \begin{bmatrix} 0 & -2 & 1 \\ 2 & 3 & -2 \\ -1 & -1 & 1 \end{bmatrix}$$

Q can also be obtained by taking the inverse of P. i.e. $Q = P^{-1}$

Part (c)

Since E is the usual basis, the change-of-basis matrix P from E to S, can be written directly from S by writing the basis vectors in S as its columns.

$$P = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 5 \\ 1 & 4 & 6 \end{bmatrix}$$

Use the definition of the change-of-basis matrix. i.e. express each vector in E as a linear combination of the vectors in S. Let $v = (a, b, c)$ be any arbitrary vector in E, then calculate the coordinates of v with respect to S as:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} y + \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} z \Longrightarrow \begin{cases} x = -2a + 2b - c \\ y = -7a + 4b - c \\ z = 5a - 3b + c \end{cases}$$

$$[v]_S = \begin{bmatrix} -2a + 2b - c \\ -7a + 4b - c \\ 5a - 3b + c \end{bmatrix}$$

Coordinates of e_i:

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 \\ -7 \\ 5 \end{bmatrix}; \; e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 \\ 4 \\ -3 \end{bmatrix}; \; e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Write the coordinates of e_i as columns of Q

$$Q = \begin{bmatrix} -2 & 2 & -1 \\ -7 & 4 & -1 \\ 5 & -3 & 1 \end{bmatrix}$$

Q can also be obtained by taking the inverse of P. i.e. $Q = P^{-1}$

Problem 6.52

Part a

The matrix $[F]_E$ represents F in basis E. Write the coefficients of x and y in $F(x,y)$ as rows to get:

$$[F]_E = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix}$$

Transition matrix between E and S is

$$P = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, \text{ and } P^{-1} = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$$

The matrix A representing F relative to the basis S is given as:

$$A = [F]_S = P^{-1}[F]_E P = \begin{bmatrix} -23 & -39 \\ 15 & 26 \end{bmatrix}$$

Alternate Method

First find the coordinates of an arbitrary vector (a,b) relative to the basis S

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} x + \begin{bmatrix} 2 \\ 3 \end{bmatrix} y \Longrightarrow \begin{cases} x + 2y = a \\ 2x + 3y = b \end{cases} \Longrightarrow \begin{cases} x = -3a + 2b \\ y = 2a - b \end{cases}$$

Hence, $[v]_S = \begin{bmatrix} -3a + 2b \\ 2a - b \end{bmatrix}$

Substitute the values of $S = [(1,2),\ (2,3)]$ in F to get

$F(S) = [F(1,2),\ F(2,3)] = [(7,-1),\ (13,0)]$

Using the equation of (a,b) for these two values and arranging them in columns give:

$$A = \begin{bmatrix} -23 & -39 \\ 15 & 26 \end{bmatrix}$$

Part b

Transition matrix between E and S' is:

$$P = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}, \text{ so } P^{-1} = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix}$$

$$B = [F]'_S = P^{-1}[F]_E P = \begin{bmatrix} 35 & 41 \\ -27 & -32 \end{bmatrix}$$

Part c

Same as 6.48 part (a). So, Transition matrix between S and S' is:

$$P = \begin{bmatrix} 3 & 5 \\ -1 & -2 \end{bmatrix}$$

Part d

Relationship is: $B = P^{-1}AP$

Problem 6.53

Part a

Find the coordinates of an arbitrary vector (a, b) relative to basis S

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} x + \begin{bmatrix} 2 \\ 5 \end{bmatrix} y \Longrightarrow \begin{cases} x + 2y = a \\ 3x + 5y = b \end{cases} \Longrightarrow \begin{cases} x = -5a + 2b \\ y = 3a - b \end{cases}$$

Hence, $[v]_S = \begin{bmatrix} -5a+2b \\ 3a-b \end{bmatrix}$

Using the transition matrix A to get the relative vector coordinates

$$AS = \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} -2 & -3 \\ 9 & 16 \end{bmatrix}$$

(Note You can do this separately using one vector at a time)

Now using the vector coordinates in the expression for $[v]_S$, we get:

$$B = \begin{bmatrix} 28 & 47 \\ -15 & -25 \end{bmatrix}$$

Alternate Method

Use $B = P^{-1}AP$ where P is simple change of basis matrix from E to S.

$P = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$ and $P^{-1} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$

$$B = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 28 & 47 \\ -15 & -25 \end{bmatrix}$$

Part b

Find the coordinates of an arbitrary vector (a, b) relative to basis S

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} x + \begin{bmatrix} 2 \\ 4 \end{bmatrix} y \Longrightarrow \begin{cases} x + 2y = a \\ 3x + 4y = b \end{cases} \Longrightarrow \begin{cases} x = -2a + b \\ y = \frac{3}{2}a - \frac{1}{2}b \end{cases}$$

Hence, $[v]_S = \begin{bmatrix} -2a+b \\ \frac{3}{2}a - \frac{1}{2}b \end{bmatrix}$

Using the transition matrix A to get the relative vector coordinates

$$AS = \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 9 & 14 \end{bmatrix}$$

(Note: You can do this separately using one vector at a time)

Now using the vector coordinates in the expression for $[v]_S$, we get:

$$B = \begin{bmatrix} 13 & 18 \\ -\frac{15}{2} & -10 \end{bmatrix}$$

Alternate Method

Use $B = P^{-1}AP$ where P is simple change of basis matrix from E to S.

$$P = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ and } P^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

$$B = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 13 & 18 \\ -\frac{15}{2} & -10 \end{bmatrix}$$

Problem 6.54 a

$$[F]_E = \begin{bmatrix} 1 & -3 \\ 2 & -4 \end{bmatrix}$$

Transition matrix between E and S is

$$P = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}$$

Hence, $[F]_S = P^{-1}[F]_E P = \begin{bmatrix} 43 & 60 \\ -33 & -46 \end{bmatrix}$

Problem 6.55

$$A = [A]_E = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 7 & 4 \\ 1 & 4 & 3 \end{bmatrix}$$

Transition matrix between E and S is

$$P = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$B = P^{-1}AP = \begin{bmatrix} 10 & 8 & 20 \\ 13 & 11 & 28 \\ -5 & -4 & -10 \end{bmatrix}$$

Problem 6.56

Part a

$$P^{-1} = \begin{bmatrix} 1 & -2 \\ 3 & -5 \end{bmatrix}$$

$$B = P^{-1}AP = \begin{bmatrix} -34 & 57 \\ -19 & 32 \end{bmatrix}$$

Part c

$$\det(A) = 1(-3) - 1(2) = -3 - 2 = -5$$

$$\det(B) = -34(32) - 57(-19) = -1088 + 1083 = -5$$

Problem 6.59

Part a

The transition matrix A can be obtained form the T directly.

$$A = \begin{bmatrix} 3 & 1 \\ -2 & 4 \end{bmatrix}$$

B can be obtained from P if P is known.

$$B = P^{-1}AP = \begin{bmatrix} 8 & 11 \\ -2 & -1 \end{bmatrix} \text{ (P is given in part b).}$$

Alternatively,

$$\begin{cases} u_1 + u_2 = w_1 \\ 2u_1 + 3u_2 = w_2 \end{cases} \Rightarrow \begin{cases} u_1 = 3w_1 - w_2 \\ u_2 = -2w_1 + w_2 \end{cases}$$

Now, substitute the value of u_i in the equations of $T(u_I)$

$$\begin{aligned} T(u_1) &= 3u_1 - 2u_2 \\ T(3w_1 - w_2) &= 3(3w_1 - w_2) - 2(-2w_1 + w_2) \\ 3T(w_1) - T(w_2) &= 13w_1 - 5w_2 \end{aligned}$$

Similarly,

$$-2T(w_1) + T(w_2) = -5w_1 + 3w_2$$

Solve for $T(w_1)$ and $T(w_2)$, yield:

$$\begin{cases} T(w_1) = 8w_1 - 2w_2 \\ T(w_2) = 11w_1 - w_2 \end{cases} \Rightarrow B = \begin{bmatrix} 8 & 11 \\ -2 & -1 \end{bmatrix}$$

Part b

The transition matrix between S and S' can satisfy the relationship:

$B = P^{-1}AP$, so it is:

$$P = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$$