Linear Algebra, Spring 2005

Solutions

May 4, 2005

Problem 5.56-5.58(a)

See Solutions for assigned problems 7.

Problem 5.76

(a)

Find $\ker F$ by setting $F(v) = \mathbf{0}$, where $v = (x, y, z)$

$$(x+y+z,\ 2x+3y+5z,\ x+3y+7z) = \mathbf{0} \Longrightarrow \begin{cases} x+y+z=0 \\ 2x+3y+5z=0 \\ x+3y+7z=0 \end{cases}$$

We row-reduce the corresponding matrix of coefficients:

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 1 & 3 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \sim$$

The mapping is singular, as seen by the last row of zeros (rank = 2).

$$\begin{cases} x+y+z=0 \\ 2x+3y+5z=0 \\ x+3y+7z=0 \end{cases} \Longleftrightarrow \begin{cases} x+y+z=0 \\ y+3z=0 \end{cases}$$

z is a free variable. Then $y = -3z$ and $x = 2z$, hence the solution is $(2z, -3z, z) = z(2, -3, 1)$. A non-zero vector in the kernel would be, for instance, $(2, -3, 1)$.

(b)

Find $\ker G$ by setting $G(v) = 0$, where $v = (x, y, z)$

$$\begin{cases} x + y \qquad\quad = 0 \\ x + 2y + 2z = 0 \\ \qquad\ y + z = 0 \end{cases}$$

We row-reduce the corresponding matrix of coefficients:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 2 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix}$$

The rank is 3 so the mapping is non-singular.

$$\begin{cases} x + y = 0 \\ x + 2y + 2z = 0 \\ y + z = 0 \end{cases} \iff \begin{cases} x + y = 0 \\ y + 2z = 0 \\ -z = 0 \end{cases}$$

The only solution is $x = 0, y = 0, z = 0$, hence G is nonsingular
For G^{-1} invert the matrix:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 2 \\ 0 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 & -2 \\ 1 & -1 & 2 \\ -1 & 1 & -1 \end{bmatrix}$$

Hence, $G^{-1}(a, b, c) = (b - 2c,\ a - b + 2c,\ -a + b - c)$

(c)

Find $\ker H$ by setting $H(v) = 0$, where $v = (x, y)$

$$\begin{cases} x + 2y = 0 \\ x - y = 0 \\ x + y = 0 \end{cases}$$

The only solution is $x = 0, y = 0$, hence H is non-singular. But H is NOT invertible, because $\dim P_2(t) = 3 > \dim R^2 = 2$, i.e. the matrix defining H is not square.

Problem 5.79

(a)

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$$

$$\begin{cases} x + 2y = 0 \\ 2x + 3y = 0 \end{cases} \Rightarrow \begin{cases} x + 2y = 0 \\ -y = 0 \end{cases}$$

The only solution is $x = 0, y = 0$, hence T is non-singular.

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$$

Hence, $T^{-1}(a, b) = (-3a + 2b,\ 2a - b)$

(b)

$$\begin{bmatrix} 2 & -3 \\ 3 & -4 \end{bmatrix} \sim \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix}$$

$$\begin{cases} 2x - 3y = 0 \\ 3x - 4y = 0 \end{cases} \Rightarrow \begin{cases} 2x - 3y = 0 \\ y = 0 \end{cases}$$

The only solution is $x = 0, y = 0$, hence T is non-singular.

$$\begin{bmatrix} 2 & -3 \\ 3 & -4 \end{bmatrix}^{-1} = \begin{bmatrix} -4 & 3 \\ -3 & 2 \end{bmatrix}$$

Hence, $T^{-1}(a, b) = (-4a + 3b,\ -3a + 2b)$

5.80

(a)

$$\begin{cases} x - 3y - 2z = 0 \\ \qquad y - 4z = 0 \\ \qquad\qquad z = 0 \end{cases} \Longrightarrow \begin{cases} x = 0 \\ y = 0 \\ z = 0 \end{cases}$$

Hence, T) is nonsingular.

$$\begin{bmatrix} 1 & -3 & -2 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 3 & 14 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

$T^{-1}(a, b, c) = (a + 3b + 14c,\ \ b + 4c,\ \ c)$

(b)

$$\begin{cases} x \qquad + z = 0 \\ x - y \qquad = 0 \\ \quad y \qquad = 0 \end{cases} \Longrightarrow \begin{cases} x = 0 \\ y = 0 \\ z = 0 \end{cases}$$

Hence, $T(x, y, z)$ is nonsingular.

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & -1 \end{bmatrix}$$

$T^{-1}(a, b, c) = (b + c,\ c,\ \ a - b -\ c)$

Problem 6.37

$F(x, y) = (4x + 5y,\ 2x - y)$

(a) $A = [F]_E = \begin{bmatrix} 4 & 5 \\ 2 & -1 \end{bmatrix}$.

(b) Basis: $u_1 = (1,4)$, $u_2 = (2,9)$

Follow the steps on slide 50 (finding a matrix representation):

Step 0.convenience $\begin{bmatrix} a \\ b \end{bmatrix} = x \cdot u_1 + y \cdot u_2 = x \cdot \begin{bmatrix} 1 \\ 4 \end{bmatrix} + y \cdot \begin{bmatrix} 2 \\ 9 \end{bmatrix}$

Solve for x, y in terms of a, b. We have

$$\begin{cases} x = 9a - 2b \\ y = b - 4a \end{cases}$$

Step 1.

$F(u_1) = F(1,4) = F(4(1) + 5(4),\ \ 2(1) - 1(4)) = (24, -2) = 220u_1 - 98u_2$ (the last equality is determined by Step 0).

$F(u_2) = F(2,9) = (53, -5) = 487u_1 - 217u_2$

Step 2.

$$B = [F]_S = \begin{bmatrix} 220 & 487 \\ -98 & -217 \end{bmatrix}$$

(c) Transition matrix P between E and S is

$$P = \begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix} (= \begin{bmatrix} u_1 & u_2 \end{bmatrix}), \text{ and } P^{-1} = \begin{bmatrix} 9 & -2 \\ -4 & 1 \end{bmatrix}$$

such a P satisfies $B = P^{-1}AP$.

(d)

For any vector $v = (a,b)$, $[v]_S = [9a - 2b, b - 4a]^T$ (from Step 0 in part (b))

$F(v) = (4a + 5b, 2a - b)$, so we have

$[F(v)]_S = [9(4a+5b) - 2(2a-b), (2a-b) - 4(4a+5b)]^T = [32a + 47b, -14a - 21b]^T$

Note that $[F]_S$ is given in part (b) as:

$$[F]_S = B = \begin{bmatrix} 220 & 487 \\ -98 & -217 \end{bmatrix}$$

Hence, we obtain:

$$[F]_S \cdot [v]_S = \begin{bmatrix} 220 & 487 \\ -98 & -217 \end{bmatrix} \cdot \begin{bmatrix} 9a - 2b \\ b - 4a \end{bmatrix} = \begin{bmatrix} 32a + 47b \\ -14a - 21b \end{bmatrix} = [F(v)]_S$$

Problem6.38

$A = [A]_E$

(a)

we use 6.37(a):

$$P = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} 4 & -1 \\ -\frac{3}{2} & \frac{1}{2} \end{bmatrix}$$

then $B = [A]_S = P^{-1}AP = \begin{bmatrix} -6 & -28 \\ 4 & 15 \end{bmatrix}$

(b)

$[v]_S = \begin{bmatrix} x \\ y \end{bmatrix}$ such that

$$\begin{bmatrix} a \\ b \end{bmatrix} = x \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} + y \cdot \begin{bmatrix} 2 \\ 8 \end{bmatrix} \Longrightarrow \begin{cases} x + 2y = a \\ 3x + 8y = b \end{cases}$$

Solve for x, y,

$[v]_S = \begin{bmatrix} 4a - b \\ \frac{3}{2}a + \frac{1}{2}b \end{bmatrix}$

$[A(v)]_S = [A]_S[v]_S = \begin{bmatrix} 18a - 8b \\ -\frac{13}{2}a + \frac{7}{2}b \end{bmatrix}$

Problem 6.39

(a)

We find the image of the standard basis vectors E:

$L(1,0) = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$
$L(0,1) = (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$

$[L]_E = \frac{1}{2}\begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix}$

(b)

We find the image of usual basis E:

$L(1,0) = (0,1)$
$L(0,1) = (1,0)$

$[L]_E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(c)

We are given the image of usual basis E:

$L(1,0) = (3,5)$
$L(0,1) = (7,-2)$

$[L]_E = \begin{bmatrix} 3 & 7 \\ 5 & -2 \end{bmatrix}$

(d)

We're given a non-standard basis $S = \{(1,1),(1,2)\}$. The change of basis matrix for S is

$$P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

with inverse

$$P^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

$$L\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix} = -1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 4\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

and

$$L\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \end{bmatrix} = 14\begin{bmatrix} 1 \\ 1 \end{bmatrix} - 9\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

To find $[L]_S$ we had to convert the image vectors into the S coordinates above, using the matrix P^{-1} [equivalent to our convenience step]. From the results above we get:

$$[L]_S = \begin{bmatrix} -1 & 14 \\ 4 & -9 \end{bmatrix}$$

To get $[L]_E$, the matrix representation with respect to the standard basis, we use the relationship that $[L]_S = P^{-1}[L]_E P$. So $[L]_E$ is given by:

$$[L]_E = P[L]_S P^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -1 & 14 \\ 4 & -9 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 7 & -4 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 18 & -11 \end{bmatrix}$$

Note: We weren't actually asked for $[L]_S$ is this problem, so you could have saved effort by observing that we didn't really need to find the S coordinates of the image vectors. You could simply have worked with the matrix $P[L]_S$, which appears in the second last line of above. In effect we calculated

$$P \cdot P^{-1}\begin{bmatrix} 3 & 5 \\ 7 & -4 \end{bmatrix} \cdot P^{-1}$$

when we could have calculated directly:

$$\begin{bmatrix} 3 & 5 \\ 7 & -4 \end{bmatrix} \cdot P^{-1}$$

which has two less matrix products.

Problem 6.40

(a)

$$[A]_E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(b)

$$[A]_E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

(c)

$$[A]_E = \begin{bmatrix} 2 & -7 & -4 \\ 3 & 1 & 4 \\ 6 & -8 & 1 \end{bmatrix}$$

Problem 6.41

We use 6.37(c):

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 3 & 5 \end{bmatrix}, P^{-1} = \begin{bmatrix} -1 & 2 & -1 \\ 5 & -5 & 2 \\ -3 & 3 & -1 \end{bmatrix}$$

(a)

$$A = [A]_E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B = P^{-1}AP = \begin{bmatrix} 1 & 3 & 5 \\ 0 & -5 & -10 \\ 0 & 3 & 6 \end{bmatrix}$$

(b)

$$A = [A]_E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$B = P^{-1}AP = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 2 & 3 \\ 1 & 0 & 0 \end{bmatrix}$$

(c)

$$A = [A]_E = \begin{bmatrix} 2 & -7 & -4 \\ 3 & 1 & 4 \\ 6 & -8 & 1 \end{bmatrix}$$

$$B = P^{-1}AP = \begin{bmatrix} 15 & 65 & 104 \\ -49 & -219 & -351 \\ 29 & 130 & 208 \end{bmatrix}$$

Problem 6.42

(a)

$$A = [L]_E = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 2 \\ 1 & 5 & 2 \end{bmatrix}$$

(b)

Since the basis $S = \{(1,1,0),(1,2,3),(1,3,5)\}$, the transition matrix between E and S is:

$$P = [\ u_1 \quad u_2 \quad u_3\ ] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 3 & 5 \end{bmatrix}, \text{ and } P^{-1} = \begin{bmatrix} -1 & 2 & -1 \\ 5 & -5 & 2 \\ -3 & 3 & -1 \end{bmatrix}$$

and hence, $B = [L]_S = P^{-1}[L]_E P = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 14 & 22 \\ 0 & -5 & -8 \end{bmatrix}$

Note: The method of slide 50 (or algorithm 6.1 in the text) can also be employed to solve part b, and the answer in the textbook is not correct.