Linear Algebra, Spring 2005

Solutions

May 4, 2005

Problem 4.89

To check for linear independence write the vectors as rows of a matrix. Reduce the matrix to echelon form and determine the number of non-zero rows (rank).

(a)

$$\begin{bmatrix} 1 & 2 & -3 & 1 \\ 3 & 7 & 1 & -2 \\ 1 & 3 & 7 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 1 & 10 & -5 \\ 0 & 1 & 10 & -5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 1 & 10 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$rank = 2 < 3$, therefore the vectors are linearly **dependent**.

(b)

$$\begin{bmatrix} 1 & 3 & 1 & -2 \\ 2 & 5 & -1 & 3 \\ 1 & 3 & 7 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 1 & -2 \\ 0 & 1 & 3 & -7 \\ 0 & 0 & -6 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 1 & -2 \\ 0 & 1 & 3 & -7 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$rank = 3$, therefore the vectors are linearly **independent**.

Problem 4.90

To solve this, form the vectors u, v and w as a linear combination:

$a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = \mathbf{0}$

If the only answer to this is $a = b = c = 0$, then the polynomials are linearly independent.

(a)

$a(t^3 - 4t^2 + 3t + 3) + b(t^3 + 2t^2 + 4t - 1) + c(2t^3 - t^2 - 3t + 5) = 0$

Arranging the terms of t,

$(a+b+2c)t^3+(-4a+2b-c)t^2+(3a+4b-3c)t+(3a-b+5c) = 0t^3+0t^2+0t+0 = 0$

This is an identity valid for all t, the coefficients of powers of t on left and right sides must be equal:

$$\begin{aligned} a + b + 2c &= 0 \\ -4a + 2b - c &= 0 \\ 3a + 4b - 3c &= 0 \\ 3a - b + 5c &= 0 \end{aligned}$$

The coefficient matrix of this system is

$$\begin{bmatrix} 1 & 1 & 2 \\ -4 & 2 & -1 \\ 3 & 4 & -3 \\ 3 & -1 & 5 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now reducing the matrix to echelon form:

$$\begin{bmatrix} 1 & 1 & 2 \\ -4 & 2 & -1 \\ 3 & 4 & -3 \\ 3 & -1 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 6 & 7 \\ 0 & -1 & 9 \\ 0 & 4 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 6 & 7 \\ 0 & 0 & 61 \\ 0 & 0 & 22 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 6 & 7 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore

$$\begin{aligned} c &= 0 \\ 6b + 7c &= 0 \rightarrow 6b = 0 \rightarrow b = 0 \\ a + b + 2c &= 0 \rightarrow a = 0 \end{aligned}$$

Conclusions: Since $a = b = c = 0$, the polynomials are linearly independent

(b)

As in (a), after writing the linear combination (scalars a, b, c) and setting equal to the zero function, you get a coefficient matrix (which can be found easily becuase it's just obtained by arranging the coefficients of each polynomial as the columns of a matrix):

$$\begin{bmatrix} 1 & 1 & 2 \\ -5 & -4 & -7 \\ -2 & -3 & -7 \\ 3 & 4 & 9 \end{bmatrix}$$

Now row reduce the matrix to echelon form:

$$\begin{bmatrix} 1 & 1 & 2 \\ -5 & -4 & -7 \\ -2 & -3 & -7 \\ 3 & 4 & 9 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \\ 0 & -1 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The last two rows have all entries equal to zero. From the first two rows we can write the expressions for a, b as functions of c

$b + 3c = 0 \rightarrow b = -3c$

$a + b + 2c = 0 \rightarrow a - 3c + 2c = 0 \rightarrow a = c$

Hence, c can be any real number, $b = -3c, a = c$. Since there are non-zero solutions for the scalars in the linear combination the polynomials are linearly dependent.

Alternate Solution, use coordinates:

For example for part a) write the polynomial coordinate vectors with respect to the standard mononomial basis $\{t^3, t^2, t, 1\}$ for P_3 space:

$[1, -4, 3, 3], [1, 2, 4, -1], [2, -1, -3, 5]$

Now check whether these vectors are linearly independent as vectors in R^4 using the usual method. For instance, write them as rows of a matrix and row reduce. Check the rank of the echelon form. Obviously the calculations will be identical to those shown above.

Problem 4.91

(a)

To check the functions $f(t) = e^t, g(t) = \sin t, h(t) = t^2$ for linear independence, write a linear combination of these functions and equate it to zero function $\mathbf{0}$ using unknown scalars a, b, c:

$af + bg + ch = \mathbf{0}$

This is an equation involving equality of functions as vectors, therefore it must be valid point-wise for ANY value of t (*this is how the vector space operations are defined for function spaces*). In school math parlance this is called an identity:

$af(t) + bg(t) + ch(t) = 0$ for all t

To show the independence show that $a = b = c = 0$. Selecting specific values of t and then solving the following equation will solve/eliminate the values of a, b, c.

$ae^t + b\text{sin}t + ct^2 = 0$

Set $t = 0 \rightarrow a(e^0) + b(\sin 0) + c(0^2) = 0 \rightarrow a(1) = 0 \rightarrow a = 0$

So now we have $b \sin t + ct^2 = 0$.

Set $t = \pi \rightarrow c\pi^2 = 0 \rightarrow c = 0$.

We are left with $b\text{sin } t = 0$ for any t, so we must have $b = 0$ as well.

Consequently the three functions are linearly independent because only the zero linear combination equals the zero function.

(b)

To check the functions $f(t) = e^t, g(t) = e^{2t}, h(t) = t$ for linear independence write a linear combination and equate to zero vector (zero function $\mathbf{0}$) using unknown scalars a, b, c:

$af + bg + ch = \mathbf{0}$

This must be valid point-wise for ANY value of t, giving the identity:

$af(t) + bg(t) + ch(t) = 0$ for all t

We can select specific values of t to solve/eliminate the values a, b, c and show that they must be zero.

Set $t = 0 \to a(e^0) + b(e^0) + c(0) = 0 \to a + b = 0 \to a = -b$

Set $t = 1 \to a(e^1) - a(e^2) + c(1) = 0 \to a(e - e^2) + c = 0 \cdots (1)$

Set $t = -1 \to a(e^{-1}) - a(e^{-2}) + c(-1) = 0 \to a(e^{-1} - e^{-2}) - c = 0 \cdots (2)$

Add equations (1) & (2): $a(e - e^2 + e^{-1} - e^{-2}) = 0$

The value in the parenthesis is not zero, therefore we must have $a = 0$ and since $a = -b$, therefore $b = 0$.

Now go back to equation (1) and conclude that $c = 0$.

Consequently the three functions are linearly independent because only the zero linear combination equals the zero function.

Problem 4.93

Given u, v, w linearly independent vectors. S is a set of three vectors which is a combination of these independent vectors. To find the linear independence of S check the linear combination of the vectors in S.

4.93 a

$S = \{u + v - 2w, u - v - w, u + w\}$

Write in matrix form and convert to echelon form

$$\begin{bmatrix} 1 & 1 & -2 \\ 1 & -1 & -1 \\ 1 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -2 \\ 0 & 2 & -1 \\ 0 & 1 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & -3 \\ 0 & 2 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & -5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

As $rank = 3$, S is independent.

4.93 b

$S = \{u + v - 3w, u + 3v - w, v + w\}$

Write in matrix form and convert to echelon form

$$\begin{bmatrix} 1 & 1 & -3 \\ 1 & 3 & -1 \\ 0 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -3 \\ 0 & -2 & -2 \\ 0 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

As $rank = 2$, S is dependent. This means that one of the vectors in S can be written as a linear combination of the other two vectors, therefore S is linearly dependent.

Note: Book question is incorrect for part b.

Problem 4.97

Use the casting-out algorithm to get the solution. Each vector is written as a column in the matrix (the u's are shown just to indicate which vector each column refers to):

(a)

$$\begin{bmatrix} \bar{u}_1 & \bar{u}_2 & \bar{u}_3 & \bar{u}_4 \\ 1 & 1 & 3 & 1 \\ 1 & 2 & 5 & 2 \\ 1 & -1 & -1 & 1 \\ 2 & -2 & -2 & -1 \\ 3 & 1 & 5 & 4 \end{bmatrix} \to \begin{bmatrix} \bar{u}_1 & \bar{u}_2 & \bar{u}_3 & \bar{u}_4 \\ 1 & 1 & 3 & 1 \\ 0 & -1 & -2 & -1 \\ 0 & 2 & 4 & 0 \\ 0 & 4 & 8 & 3 \\ 0 & 2 & 4 & -1 \end{bmatrix} \to \begin{bmatrix} \bar{u}_1 & \bar{u}_2 & \bar{u}_3 & \bar{u}_4 \\ 1 & 1 & 3 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -3 \end{bmatrix} \to \begin{bmatrix} \bar{u}_1 & \bar{u}_2 & \bar{u}_3 & \bar{u}_4 \\ 1 & 1 & 3 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The columns of the original matrix which correspond to columns with pivots in the row reduced echelon form provide our basis: $W = \text{span}(\bar{u}_1, \bar{u}_2, \bar{u}_4)$ and $\{u_1, u_2, u_4\}$ is a basis for W. NOTE: Book answer is incorrect for this question.

(b)

$$\begin{bmatrix} \bar{u}_1 & \bar{u}_2 & \bar{u}_3 & \bar{u}_4 \\ 1 & -2 & 1 & 3 \\ -2 & 4 & -3 & -7 \\ 1 & -2 & 1 & 3 \\ 3 & -6 & 2 & 8 \\ -1 & 2 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} \bar{u}_1 & \bar{u}_2 & \bar{u}_3 & \bar{u}_4 \\ 1 & -2 & 1 & 3 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} \bar{u}_1 & \bar{u}_2 & \bar{u}_3 & \bar{u}_4 \\ 1 & -2 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The columns of the original matrix which correspond to columns with pivots in the row reduced echelon form provide our basis: $W = \text{span}(\bar{u}_1, \bar{u}_3)$ and a required basis is $\{u_1, u_3\}$. NOTE: Book answer is incorrect for this question.

(c)

$$\begin{bmatrix} \bar{u}_1 & \bar{u}_2 & \bar{u}_3 & \bar{u}_4 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} \bar{u}_1 & \bar{u}_2 & \bar{u}_3 & \bar{u}_4 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & -2 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \xrightarrow[swap(2,5)]{} \begin{bmatrix} \bar{u}_1 & \bar{u}_2 & \bar{u}_3 & \bar{u}_4 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 \end{bmatrix}$$

$$\to \begin{bmatrix} \bar{u}_1 & \bar{u}_2 & \bar{u}_3 & \bar{u}_4 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} \bar{u}_1 & \bar{u}_2 & \bar{u}_3 & \bar{u}_4 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} \bar{u}_1 & \bar{u}_2 & \bar{u}_3 & \bar{u}_4 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The columns of the original matrix which correspond to columns with pivots in the row reduced echelon form provide our basis: $W = \text{span}(\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_4)$ and the basis for W is $\{u_1, u_2, u_3, u_4\}$. The four vectors are linearly independent.

(d)

$$\begin{bmatrix} \bar{u}_1 & \bar{u}_2 & \bar{u}_3 & \bar{u}_4 \\ 1 & 2 & 1 & 4 \\ 0 & 1 & 1 & 2 \\ 1 & 2 & 2 & 5 \\ 1 & 0 & 3 & 4 \\ 1 & 1 & 4 & 6 \end{bmatrix} \to \begin{bmatrix} \bar{u}_1 & \bar{u}_2 & \bar{u}_3 & \bar{u}_4 \\ 1 & 2 & 1 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -1 & -1 \\ 0 & 2 & -2 & 0 \\ 0 & 1 & -3 & -2 \end{bmatrix} \to \begin{bmatrix} \bar{u}_1 & \bar{u}_2 & \bar{u}_3 & \bar{u}_4 \\ 1 & 2 & 1 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 4 & 4 \end{bmatrix} \to \begin{bmatrix} \bar{u}_1 & \bar{u}_2 & \bar{u}_3 & \bar{u}_4 \\ 1 & 2 & 1 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The columns of the original matrix which correspond to columns with pivots in the row reduced echelon form provide our basis: $W = \text{span}(\bar{u}_1, \bar{u}_2, \bar{u}_3)$ and the basis for W is $\{u_1, u_2, u_3\}$.

Problem 4.98

4.98 a

$U = \{(a, b, c, d) : b - 2c + d = 0\}$

The constraint that defines U gives the form of any vector (a, b, c, d) in U: a can be any real number; b, c, and d are constrained by the equation. If we allow b and c to be any real number, then d has to be: $d = 2c - b$ We can therefore express any vector in U in the form:

$(a, b, c, 2c - b) = a(1, 0, 0, 0) + b(0, 1, 0, -1) + c(0, 0, 1, 2)$

i.e. a linear combination of the three vectors

$\{(1, 0, 0, 0), (0, 1, 0, -1), (0, 0, 1, 2)\}$.

These three vectors are linearly independent [for completeness, e.g. on a test, this ought to be checked following the method as in problem 4.89 above]. Therefore these three vectors provide a basis for U and so U is 3 dimensional.

4.98 b

$W = \{(a, b, c, d) : a = d, b = 2c\}$

The constraint which defines W provides the form for any vector in W: let a and b be any real number, then c and d are fixed by the constraint: $a \in R, b \in R, c = \frac{1}{2}b, d = a$

So any vector in W can be expressed in the form:

$(a, 2b, b, a) = a(1, 0, 0, 1) + b(0, 2, 1, 0)$

i.e. a linear combination of the two vectors $\{(1, 0, 0, 1), (0, 2, 1, 0)\}$. These two vectors are linearly independent because one is not a scalar multiple of the other [that argument is sufficient, no need to do any calculations as in problem 4.89 when you have two vectors]. Therefore these two vectors provide a basis for W and so W is 2 dimensional.

4.98 c

$U \cap W = \{(a, b, c, d) : b - 2c + d = 0, a = d, b = 2c\}$

To follow the same approach as above with three constraints like this, you need: (i) to check if the three constraints are independent, and (ii) express them in a 'row reduced echelon form' so you can write the expression for your general vector in the subspace. This isn't so easy to do by inspection, so write out the constraints as three equations:

$b - 2c + d = 0$

$a - d = 0$

$b - 2c = 0$

Now row reduce these to echelon form [use a matrix for convenience]:

$$\begin{bmatrix} 0 & 1 & -2 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 1 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This gives the required form of the constraints:

$a - d = 0, b - 2c + d = 0, d = 0$

to express the general form of a vector in $U \cap W$ by inspection:

$(0, 2c, c, 0) = c(0, 2, 1, 0)$

So all vectors in the intersection space are multiples of the vector $(0, 2, 1, 0)$ and $\{(0, 2, 1, 0)\}$ is therefore a basis. The space is one-dimensional.

Another simpler approach: a) U, a subspace of R^4, is defined by one constraint [which must be independent], therefore $\dim U = 4 - 1 = 3$; b) W is defined by two constraints, which you could show to be independent simply by arguing that the one is not a multiple of the other], therefore $\dim W = 4-2 = 2$; c) $U \cap W$ is defined by three independent constraints, as shown in c) part above, there $\dim(U \cap W) = 4 - 3 = 1$.

Yet another variation, using the later concept of vector space sums: a) and b) as above. Then combine the three basis vectors you got for U in

a) and the two you got for U in b) and check linear independence for the five combined vectors, which span $U + W = \text{span}\{U, W\}$. You will see that four of these are linearly independent and so $U + W = R^4$ and $\dim(U + W) = 4$. Then conclude:

$\dim(U \cap W) = \dim U + \dim W - \dim(U + W) = 3 + 2 - 4 = 1.$

Obviously the approach taken provides different amounts of work. You should choose the method based on what the question actually asks for, e.g. does it ask for bases for the subspaces. If not there is no need to find bases, unless it provides the easiest method anyway. The most direct approach for this question would probably be the "another simpler approach" described above.

Problem 4.101

4.101 a

Write the polynomials as coordinate vectors with respect to the standard mononomial basis for P_n and arrange these as the row of a matrix:

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Here we have used the basis in the following order: $\{t^n, t^{n-1}, t^{n-2}, ..., t^1, 1\}$. The order is convenient because we automatically get the matrix of coordinate vectors in echelon form. We can see by inspection that there are n independent rows, and therefore the coordinate vectors are linearly independent. n independent coordinate vectors in R^n must automatically be a basis. Finally, we conclude [by isomorphism] that the original polynomial vectors must also be a basis for P_n.

Note the usefulness of the idea of coordinate vectors in this problem. Also, had we written the mononomial basis in the other order, say, we would have got a matrix of coordinate vectors that wasn't automatically in echelon form. We would have had to row reduce it. So choosing the order carefully for the vectors in the coordinate basis can save work.

4.101 b

The dimension of P_n is $n+1$. The given set of vectors has only n polynomials in it. Therefore it cannot be a basis.

Problem 4.102

To solve these, we can use either the row space method or the casting-out algorithm with columns. Since we used casting-out earlier, and since the question does not ask for a subset of the original vectors, we'll use the row space method to solve this problem. There is a typo in question 4.102 a) after some reverse engineering, I think $v = t^3 + 3t^2 - t + 4$ was intended. We'll assume that the problem meant that.

4.102 a

Write the polynomials as coordinate vectors using the standard mononomial basis and arrange as rows of a matrix.

$$\begin{bmatrix} \bar{u} & 1 & 2 & -2 & 1 \\ \bar{v} & 1 & 3 & -1 & 4 \\ \bar{w} & 2 & 1 & -7 & -7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -2 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & 3 & 3 & 9 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -2 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The rank of the row reduced echelon matrix in 2 [two non-zero rows]. Therefore, $\dim(W) = 2$.

A basis is found by using the coefficients of the first two rows:

$\{[1, 2, -2, 1], [0, 1, 3, 3]\}$

corresponding to the polynomial basis:

$\{t^3 + 2t^2 - 2t + 1, t^2 + 3t + 3\}$

4.102 b

$$\begin{bmatrix} \bar{u} & 1 & 1 & -3 & 2 \\ \bar{v} & 2 & 1 & 1 & -4 \\ \bar{w} & 4 & 3 & -5 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -3 & 2 \\ 0 & 1 & -7 & 8 \\ 0 & -1 & -7 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -3 & 2 \\ 0 & 1 & -7 & 8 \\ 0 & 0 & -14 & 14 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -3 & 2 \\ 0 & 1 & -7 & 8 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Rank is 3, therefore $\dim(W) = 3$.

A basis is found by using the coefficients of the three rows:

$\{t^3 + t^2 - 3t + 2, t^2 - 7t + 8, t - 1\}$

Problem 4.103

To solve this directly take a linear combination of the four matrices.

$aA + bB + cC + dD = 0$

$$a \begin{bmatrix} 1 & -5 \\ -4 & 2 \end{bmatrix} + b \begin{bmatrix} 1 & 1 \\ -1 & 5 \end{bmatrix} + c \begin{bmatrix} 2 & -4 \\ -5 & 7 \end{bmatrix} + d \begin{bmatrix} 1 & -7 \\ -5 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} a + b + 2c + d & -5a + b - 4c - 7d \\ -4a - b - 5c - 5d & 2a + 5b + 7c + d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Equating entries in the matrices you get four equations in the four scalars a, b, c, d, a system with zero righthand-side and coefficient matrix:

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ -5 & 1 & -4 & -7 \\ -4 & -1 & -5 & -5 \\ 2 & 5 & 7 & 1 \end{bmatrix}$$

[Alternatively we can express the four matrices as coordinate vectors with respect to the standard basis for

$M_{2,1}: [1,-5,-4,2],[1,1,-1,5],[2,-4,-5,7],[1,-7,-5,1]$

and check this set of vectors in R^4 for linear independence using the casting out algorithm and columns. You arrive at the same point this way, but more quickly.]

Now row reduce the matrix to echelon form:

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ -5 & 1 & -4 & -7 \\ -4 & -1 & -5 & -5 \\ 2 & 5 & 7 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 6 & 6 & -2 \\ 0 & 3 & 3 & -1 \\ 0 & -3 & -3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 3 & 3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The rank is two. Therefore $\dim(W) = 2$.

Continuing the casting out algorithm we would select the original vectors corresponding to columns in the reduced matrix that don't have pivots, i.e. throw away the vector in column 3. So the required basis is $\{A, B, D\}$.

Note: This basis consists of matrices from the original set of given matrices, which is not strictly required for this problem (it doesn't ask for that). You could have used an alternative algorithm (e.g. row method) in this case. The above solution is not difficult though.