Linear Algebra, Spring 2005

Solutions

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Problem 4.75

To show that V is a vector space over K, all axioms must be satisfied.

Let $a = (a_1, a_2, ...)$, $b = (b_1, b_2, ...)$ and $c = (c_1, c_2, ...)$

$a, b, c \in V$ and $k, l \in K$

Vector addition axioms

- Closure

  $a + b = (a_1, a_2, ...) + (b_1, b_2, ...) = (a_1 + b_1, a_2 + b_2, ...)$

  this is of the form of vector V, hence closure is satisfied.

- Commutative

  $a + b = (a_1, a_2, ...) + (b_1, b_2, ...) = (a_1 + b_1, a_2 + b_2, ...)$

  $b + a = (b_1, b_2, ...) + (a_1, a_2, ...) = (b_1 + a_1, b_2 + a_2, ...)$

  Clearly $a + b = b + a$, so axiom satisfied.

- Associative

$$\begin{aligned} a + (b + c) &= (a_1, a_2, ...) + ((b_1, b_2, ...) + (c_1, c_2, ...)) \\ &= (a_1, a_2, ...) + (b_1 + c_1, b_2 + c_2, ...) = (a_1 + b_1 + c_1, a_2 + b_2 + c_2, ...) \\ (a + b) + c &= ((a_1, a_2, ...) + (b_1, b_2, ...)) + (c_1, c_2, ...) \end{aligned}$$

$$
\begin{aligned}
&= (a_1+b_1, a_2+b_2, ...) + (c_1, c_2, ...) = (a_1+b_1+c_1, a_2+b_2+c_2, ...) \\
a+(b+c) &= (a+b)+c
\end{aligned}
$$

- Zero vector

$$
\begin{aligned}
a+\mathbf{0} &= (a_1, a_2, ...) + (0, 0, ...) = (a_1, a_2, ...) = a \\
\mathbf{0}+a &= (0, 0, ...) + (a_1, a_2, ...) = (a_1, a_2, ...) = a
\end{aligned}
$$

- Additive inverse

$$
\begin{aligned}
a+(-a) &= (a_1, a_2, ...) + (-(a_1, a_2, ...)) = (a_1 - a_1, a_2 - a_2, ...) = (0, 0, ...) = \mathbf{0} \\
-a+a &= -(a_1, a_2, ...) + (a_1, a_2, ...) = (-a_1 + a_1, -a_2 + a_2, ...) = (0, 0, ...) = \mathbf{0}
\end{aligned}
$$

Scalar multiplication axioms

- Closure

  $ka = k(a_1, a_2, ...) = (ka_1, ka_2, ...)$

  $ka \in V$

- Vector sum distributive

$$
\begin{aligned}
k(a+b) &= k((a_1, a_2, ...) + (b_1, b_2, ...)) = k(a_1+b_1, a_2+b_2, ...) \\
&= (k(a_1+b_1), k(a_2+b_2), ...) = (ka_1+kb_1, ka_2+kb_2, ...) \\
ka+kb &= k(a_1, a_2, ...) + k(b_1, b_2, ...) \\
&= (ka_1, ka_2, ...) + (kb_1, kb_2, ...) = (ka_1+kb_1, ka_2+kb_2, ...) \\
k(a+b) &= ka+kb
\end{aligned}
$$

- Scalar sum distributive

$$
\begin{aligned}
(k+l)a &= (k+l)(a_1, a_2, ...) = ((k+l)a_1, (k+l)a_2, ...) = (ka_1+la_1, ka_2+la_2, ...) \\
ka+la &= k(a_1, a_2, ...) + l(a_1, a_2, ...) = (ka_1, ka_2, ...) + (la_1, la_2, ...) \\
&= (ka_1+la_1, ka_2+la_2, ...) \\
(k+l)a &= ka+la
\end{aligned}
$$

- Scalar multiplication associative

$$\begin{aligned} k(la) &= k(l(a_1, a_2, ...)) = k(la_1, la_2, ...) = (kla_1, kla_2, ...) \\ (kl)a &= kl(a_1, a_2, ...) = (kla_1, kla_2, ...) \\ k(la) &= (kl)a \end{aligned}$$

- Multiplication by scalar 1

$$1a = 1(a_1, a_2, ...) = (1a_1, 1a_2, ...) = (a_1, a_2, ...) = a$$

Problem 4.77

(a) $W = \{(a, b, c) \in R^3 \mid a = 3b\}$

Let $u, v \in W$ and $u = (3b_1, b_1, c_1), v = (3b_2, b_2, c_2)$ Checking for subspace conditions:

- $\mathbf{0} \in W$
- Closure under vector sum

$$\begin{aligned} u + v &= (3b_1, b_1, c_1) + (3b_2, b_2, c_2) = (3b_1 + 3b_2, b_1 + b_2, c_1 + c_2) \\ &= (3(b_1 + b_2), b_1 + b_2, c_1, c_2) \\ u + v &\in W \end{aligned}$$

- Closure under scalar multiplication

$$\begin{aligned} ku &= k(3b_1, b_1, c_1) = (k(3b_1), kb_1, kc_1) = (3kb_1, kb_1, kc_1) \\ ku &\in W \end{aligned}$$

As W satisfies all conditions it is a subspace of R^3.

(b) $W = \{(a, b, c) \in R^3 \mid a \leq b \leq c\}$

Let $u = (1, 2, 3) \in W$ Checking for closure under scalar multiplication, by taking $k = -1 \in R$:

$$\begin{aligned} ku &= -1(1, 2, 3) = (-1, -2, -3) \\ ku &\notin W \end{aligned}$$

Therefore W is not a subspace. Note: To prove that a subset is not a subspace only one example where any of the conditions fail is sufficient.

(c) $W = \{(a, b, c) \in R^3 \mid ab = 0\}$

Let $u, v \in W$ and $u = (0, 1, 1), v = (1, 0, 1)$

Checking for closure under vector addition

$$\begin{aligned} u + v &= (0, 1, 1) + (1, 0, 1) = (1, 1, 2) \\ u + v &\notin W \end{aligned}$$

Therefore W is not a subspace.

(d) $W = \{(a, b, c) \in R^3 \mid a + b + c = 0\}$

Let $u, v \in W$, $u = (a_1, b_1, -a_1 - b_1), v = (a_2, b_2, -a_2 - b_2)$

Checking for subspace conditions:

- $\mathbf{0} \in W$
- Closure under vector sum

$$\begin{aligned} u + v &= (a_1, b_1, -a_1 - b_1) + (a_2, b_2, -a_2 - b_2) = (a_1 + a_2, b_1 + b_2, -a_1 - b_1 - a_2 - b_2) \\ &= (a_1 + a_2, b_1 + b_2, -a_1 - a_2 - b_1 - b_2) \\ u + v &\in W \end{aligned}$$

- Closure under scalar multiplication

$$\begin{aligned} ku &= k(a_1, b_1, -a_1 - b_1) = (ka_1, kb_1, k(-a_1 - b_1)) = (ka_1, kb_1, -ka_1 - kb_1)) \\ ku &\in W \end{aligned}$$

As W satisfies all conditions it is a subspace of R^3. This is a hyperplane (in this case a plane) in R^3 – see lecture notes.

(e) $W = \{(a, b, c) \in R^3 \mid b = a^2\}$

Be suspicious when you see squares in linear algebra. Let $u \in W$, $u = (a, a^2, c)$ from the one constraint in the rule that defines the subset W.

Let $u = (1, 1, 1) \in W$. Checking for closure for scalar multiplication, by taking $k = -1 \in R$:

$$\begin{aligned} ku &= -1(1, 1, 1) = (-1, -1, -1) \\ ku &\notin W \end{aligned}$$

As W is not closed for scalar multiplication, therefore, it is not a subspace.

(f) $W = \{(a, b, c) \in R^3 \mid a = 2b = 3c\}$

Here we have effectively two constraints on the a, b, c for the vector to be in the subset W. Vectors in W are of the form $(6b, 3b, 2b) = b(6, 3, 2)$. Consequently $W = \text{Sp}(6, 3, 2)$, i.e. all multiples of the single vector. We can conclue that W is a subspace.

Problem 4.78

Given that V is a vector space of n x n matrices. For W to be a subspace of V, W should contain zero matrix and the it should be closed under matrix addition and scalar multiplication.

(a) Symmetric ($A^T = A$)

Zero matrix is also a symmetric matrix so W containing zero matrix condition is satisfied.
Matrix addition of two symmetric matrices will result in a symmetric matrix and also if a symmetric matrix is multiplied by a scalar, the resultant matrix will still be symmetric. Hence all conditions for a subspace are satisfied hence, W is a subspace of V.

$$\begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{12} & a_{22} & ... & a_{2n} \\ \vdots & \vdots & ... & \vdots \\ a_{1n} & a_{2n} & ... & a_{nn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & ... & b_{1n} \\ b_{12} & b_{22} & ... & b_{2n} \\ \vdots & \vdots & ... & \vdots \\ b_{1n} & b_{2n} & ... & b_{nn} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & ... & a_{1n}+b_{1n} \\ a_{12}+b_{12} & a_{22}+b_{22} & ... & a_{2n}+b_{2n} \\ \vdots & \vdots & ... & \vdots \\ a_{1n}+b_{1n} & a_{2n}+b_{2n} & ... & a_{nn}+b_{nn} \end{bmatrix}$$

(b) (Upper) triangular

Matrix addition of two upper triangular matrices will result in an upper triangular matrix, also the scalar multiplication with an upper triangular matrix will result in an upper triangular matrix. Hence the matrix addition and scalar multiplication is closed for the upper triangular matrices hence they form a subspace.

$$\begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{12} & a_{22} & ... & a_{2n} \\ \vdots & \vdots & ... & \vdots \\ 0 & 0 & ... & a_{nn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & ... & b_{1n} \\ b_{12} & b_{22} & ... & b_{2n} \\ \vdots & \vdots & ... & \vdots \\ 0 & 0 & ... & b_{nn} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & ... & a_{1n}+b_{1n} \\ a_{12}+b_{12} & a_{22}+b_{22} & ... & a_{2n}+b_{2n} \\ \vdots & \vdots & ... & \vdots \\ 0 & 0 & ... & a_{nn}+b_{nn} \end{bmatrix}$$

(c) Diagonal

Diagonal matrices have entries only at the diagonal of the matrix. The matrix addition is closed for diagonal matrices and also the scalar multiplication is closed for diagonal matrices. Zero matrix can also be a part of the diagonal matrix and hence the diagonal matrices form a subspace.

$$\begin{bmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix} + \begin{bmatrix} b_{11} & 0 & \dots & 0 \\ 0 & b_{22} & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & b_{nn} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & 0 & \dots & 0 \\ 0 & a_{22}+b_{22} & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & a_{nn}+b_{nn} \end{bmatrix}$$

(d) Scalar

All conditions for subspace are satisfied. A 1 matrix is, of course, indistinguishable from a real number.

Problem 4.80

$U < V$, $W < V$, and $U \cup W < V$. To show: $U \subseteq W$ or $W \subseteq U$.

Take $u \in U$, $w \in W$, so that $w \notin U$. Need to show that $u \in W$.

$u + w \in U \cup W$ because $U \cup W$ is a subspace, so closed to sums (and each of the vectors is in the union subspace). But $u + w \notin U$ because $w \notin U$. So $u + w \in W$. Now the vector $(u + w) - w$ must be in W since W is a subspace. This is just u of course, so we've shown that $u \in W$ as required.

Problem 4.81

(a) Bounded functions

$W = \{f : R \to R | f(x)| < M \in R\}$, i.e. M is a finite number that bounds the function values $f(x)$. Take two such bounded functions f and g, with bounds M_1 and M_2 respectively. Then

- Vector Addition $|(f+g)(x)| = |f(x)+g(x)| \leq |f(x)|+|g(x)| < M_1+M_2 = M$
  i.e. $f + g$ is bounded and so $(f + g) \in W$
- Scalar Multiplication $|(kf)(x)| = |kf(x)| \leq |k||f(x)| < |k|M_1 = M$
  i.e. kf is bounded and so $kf \in W$. Since W is closed to vector sums and scalar products it is a function subspace. [Make sure you understand the reasons for each of the steps in the lines above]

(b) Even functions

$W = \{f : R \to R | f(-x) = f(x)\}$. Take two such even functions f and g. Then

- Vector Addition $(f+g)(-x) = f(-x)+g(-x) = f(x)+g(x) = (f+g)(x)$
  i.e. $(f + g)$ is also an even function so $(f + g) \in W$
- Scalar Multiplication $(kf)(-x) = kf(-x) = kf(x) = (kf)(x)$ i.e. kf is an even function and so $kf \in W$

Since W is closed to vector sums and scalar products it is a function subspace.

Problem 4.82

V is a vector space of infinite sequences $a = (a_1, a_2, a_3, ..., a_k, ...)$

(a) $W = \{a | a_1 = 0\}$

Take two vectors $u, v \in W, u = (0, a_2, a_3, ...), v = (0, b_2, b_3, ...)$. Check closure:

- Vector addition $u + v = (0, a_2 + b_2, a_3 + b_3, a_4 + b_4, ...) \in W$
- Scalar multiplication $ku = (0, ka_2, ka_3, ...) \in W$.

  So W is closed to vector sums and scalar multiples. Therefore it is a subspace.

(b) $W = \{a|$only a finite number of a_i entries are non-zero$\}$

Take two vectors $u, v \in W, u$ with k_1 entries non-zero and v with k_2 entries non-zero. The sum $u+v$ can have at most k_1+k_2 non-zero entries, i.e. a finite number, so $u+v \in W$. Similarly, the scalar product ku has the same number of non-zero entries as in u, or none if $k = 0$. In either case ku has a finite number of non-zero entries, so $ku \in W$. Since W is closed to vector sums and scalar multiples it is a subspace.

Problem 4.83

$u = (1, 2, 3)$ and $v = (2, 3, 1)$

(a) $w = (1, 3, 8)$

Let $w = xu + yv$

$$\begin{bmatrix} 1 \\ 3 \\ 8 \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + y \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \quad \text{or} \quad \begin{aligned} x + 2y &= 1 \\ 2x + 3y &= 3 \\ 3x + y &= 8 \end{aligned}$$

These equations reduce to

$$\begin{aligned} x + 2y &= 1 \\ y &= -1 \end{aligned}$$

which have solution $x = 3, y = -1$.

Hence, $w = 3u - v$ is the required linear combination.

(b) $w = (2, 4, 5)$

Let $w = xu + yv$

$$\begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + y \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \text{ or } \begin{aligned} x + 2y &= 2 \\ 2x + 3y &= 4 \\ 3x + y &= 5 \end{aligned}$$

These equations reduce to a system with an inconsistent third equation. So there is no solution for x, y and therefore the required linear combination is not possible.

(c) $w = (1, k, -2)$

Let $w = xu + yv$

$$\begin{bmatrix} 1 \\ k \\ -2 \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + y \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \text{ or } \begin{aligned} x + 2y &= 1 \\ 2x + 3y &= k \\ 3x + y &= -2 \end{aligned}$$

These equations reduce to

$$\begin{aligned} x + 2y &= 1 \\ y &= 1 \\ 0 &= 1 - k \end{aligned}$$

For consistency we require $k = 1$ [note typo in the book]. For this value of k we get $x = -1, y = 1$, and $w = -u + v$ for k=1.

(d) $w = (a, b, c)$

Let $w = xu + yv$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + y \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \text{ or } \begin{aligned} x + 2y &= a \\ 2x + 3y &= b \\ 3x + y &= c \end{aligned}$$

These equations reduce to

$$\begin{aligned} x + 2y &= a \\ y &= 2a - b \\ 0 &= 7a - 5b + c \end{aligned}$$

For consistency we require $7a - 5b + c = 0$.

Problem 4.84

$f(t) = at^2 + bt + c$ as a linear combination of polynomials $p_1 = (t-1)^2, p_2 = t - 1, p_3 = 1$

Let x, y, z be some arbitrary constants, then

$$\begin{aligned} f(t) &= xp_1(t) + yp_2(t) + zp_3(t) = at^2 + bt + c \\ &= x(t^2 - 2t + 1) + y(t - 1) + z(1) \\ &= xt^2 + (-2x + y)t + (x - y + z) \end{aligned}$$

Compared with the given equation for $f(t)$ We get:

$$\begin{aligned} a &= x \\ b &= -2x + y \\ c &= x - y + z \end{aligned}$$

Solving for x, y, z in terms of a, b, c we get:

$$\begin{aligned} x &= a \\ y &= 2a + b \\ z &= a + b + c \end{aligned}$$

So the required l.c. is $f(t) = at^2 + bt + c = a(t-1)^2 + (2a+b)(t-1) + (a+b+c)1$

Problem 4.85

$U = (a, b, 0)$ is the xy-plane and $W = \mathrm{Sp}\{(1,1,1),(1,2,3)\}$.
A vector $u = (a, b, c) \in U \cap W$ must satisfy $(a, b, c) = x(1,1,1) + y(1,2,3)$ and $c = 0$. So we get the coordinate equations:

$$\begin{aligned} x + y &= a \\ x + 2y &= b \\ x + 3y &= 0 \end{aligned}$$

These reduce to

$$\begin{aligned} x + y &= a \\ y &= b - a \\ 0 &= 2b - a \end{aligned}$$

So the consistency condition is $0 = 2b - a$, i.e. $a = 2b$. One vector which satisfies this condition is (2,1,0). This is the required vector. [Any scalar multiple of it would also work]