Linear Algebra, Spring 2005

Solutions

May 4, 2005

Solution to 3.57

(a)

$$\begin{bmatrix} 4 \\ -9 \\ 2 \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + y \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} + z \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} x+y+z \\ 2x+4y-3z \\ -x+2y+2z \end{bmatrix}$$

Therefore, the system of equations is:

$$\begin{cases} x + y + z = 4 \\ 2x + 4y - 3z = -9 \\ -x + 2y + 2z = 2 \end{cases}$$

Reduce it to echelon form:

$$\begin{cases} x + y + z = 4 \\ 2y - 5z = -17 \\ 3y + 3z = 6 \end{cases} \Rightarrow \begin{cases} x + y + z = 4 \\ 2y - 5z = -17 \\ 7z = 21 \end{cases} \Rightarrow \begin{cases} x = 2 \\ y = -1 \\ z = 3 \end{cases}$$

Thus, $v = 2u_1 - u_2 + 3u_3$.

(b)

$$\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + y \begin{bmatrix} 2 \\ 6 \\ 5 \end{bmatrix} + z \begin{bmatrix} 1 \\ 7 \\ 8 \end{bmatrix} = \begin{bmatrix} x+2y+z \\ 2x+6y+7z \\ x+5y+8z \end{bmatrix}$$

Therefore, the equation is:

$$\begin{cases} x + 2y + z = 1 \\ 2x + 6y + 7z = 3 \\ x + 5y + 8z = 2 \end{cases}$$

Reduce it to echelon form:

$$\begin{cases} x + 2y + z = 1 \\ 2y + 5z = 1 \\ 3y + 7z = 1 \end{cases} \Rightarrow \begin{cases} x + 2y + z = 1 \\ 2y + 5z = 1 \\ z = 1 \end{cases} \Rightarrow \begin{cases} x = 4 \\ y = -2 \\ z = 1 \end{cases}$$

Thus, $v = 4u_1 - 2u_2 + u_3$.

(c)

$$\begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + y \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} + z \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix} = \begin{bmatrix} x + 2y + 3z \\ x + 3y + 5z \\ 2x + 5y + 8z \end{bmatrix}$$

Therefore, the equation is:

$$\begin{cases} x + 2y + 3z = 1 \\ x + 3y + 5z = 4 \\ 2x + 5y + 8z = 6 \end{cases}$$

Reduce it to echelon form:

$$\begin{cases} x + 2y + 3z = 1 \\ y + 2z = 3 \\ y + 2z = 4 \end{cases} \Rightarrow \begin{cases} x + 2y + 3z = 1 \\ y + 2z = 3 \\ 0 = 1 \end{cases}$$

Thus, the linear combination is not possible.

Solution to 3.58

Take the dot product of pairs of vectors to get:

$u_1 \cdot u_2 = 1(1) + 1(3) + 2(-2) = 0$, $u_1 \cdot u_3 = 1(4) + 1(-2) + 2(-1) = 0$, $u_2 \cdot u_3 = 1(4) + 3(-2) - 2(-1) = 0$.

Thus the three vectors in R^3 are orthogonal, and hence Fourier coefficients can be used to get the coefficients for the linear combination. That is, $v = xu_1 + yu_2 + zu_3$, where

$x = \frac{v \cdot u_1}{u_1 \cdot u_1}$, $y = \frac{v \cdot u_2}{u_2 \cdot u_2}$, $z = \frac{v \cdot u_3}{u_3 \cdot u_3}$

(a)

$x = \frac{5 - 5 + 18}{1 + 1 + 4} = 3$, $y = \frac{5 - 15 - 18}{1 + 9 + 4} = -2$, $z = \frac{20 + 10 - 9}{16 + 4 + 1} = 1$

Thus $v = 3u_1 - 2u_2 + u_3$

(b)

$x = \frac{1-3+6}{1+1+4} = \frac{2}{3}$, $y = \frac{1-9-6}{1+9+4} = -1$, $z = \frac{4+6-3}{16+4+1} = \frac{1}{3}$
Thus $v = \frac{2}{3}u_1 - u_2 + \frac{1}{3}u_3$

(c)

$x = \frac{1+1+2}{1+1+4} = \frac{2}{3}$, $y = \frac{1+3-2}{1+9+4} = \frac{1}{7}$, $z = \frac{4-2-1}{16+4+1} = \frac{1}{21}$
Thus $v = \frac{2}{3}u_1 + \frac{1}{7}u_2 + \frac{1}{21}u_3$

Solution to 3.59

(a)

$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ 5 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} z &= \text{free} \\ y &= z \\ x &= -2z + y = -2z + z = -z \end{aligned}$$

Solution space is one-dimensional given by $(x, y, z) = z(-1, 1, 1)$. A basis for the solution space consists of $\{u_1\}$ where $u_1 = (-1, 1, 1)$.

(b)

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 5 & 2 \\ 3 & -1 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 8 \\ 0 & 7 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 8 \\ 0 & 0 & -61 \end{bmatrix}$$

The system has only the trivial solution $x = y = z = 0$. Consequently the solution space has no basis.

(c)

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 4 & 7 & 4 \\ 3 & 6 & 10 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned}
t &= \text{free} \\
z &= -2t \\
y &= \text{free} \\
x &= -t-3z-2y = -t-3(-2t)-2y = -t+6t-2y = 5t-2y
\end{aligned}$$

The solution space is two-dimensional given by $(x, y, z, t) = (5t-2y, y, -2t, t) = y(-2, 1, 0, 0) + t(5, 0, -2, 1)$. A basis for the solution space consists of $\{u_1, u_2\}$ where $u_1 = (-2, 1, 0, 0)$ and $u_2 = (5, 0, -2, 1)$.

Solution to 3.60

(a)

$$\begin{bmatrix} 1 & 3 & 2 & -1 & -1 \\ 2 & 6 & 5 & 1 & -1 \\ 5 & 15 & 12 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 2 & -1 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 2 & -1 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Using variable names (x, y, z, s, t) for the solution we get:

$$\begin{aligned}
t &= \text{free} \\
s &= \text{free} \\
z &= -t-3s \\
y &= \text{free} \\
x &= t+s-2z-3y = t+s-2(-t-3s)-3y = 3t+7s-3y
\end{aligned}$$

The solution space is three-dimensional given by $(x, y, z, s, t) = (-3y+7s+3t, y, -3s-t, s, t) = y(-3, 1, 0, 0, 0) + s(7, 0, -3, 1, 0) + t(3, 0, -1, 0, 1)$. A basis for the solution space consists of $\{u_1, u_2, u_3\}$ where $u_1 = (-3, 1, 0, 0, 0)$, $u_2 = (7, 0, -3, 1, 0)$, and $u_3 = (3, 0, -1, 0, 1)$.

(b)

$$\begin{bmatrix} 2 & -4 & 3 & -1 & 2 \\ 3 & -6 & 5 & -2 & 4 \\ 5 & -10 & 7 & -3 & 4 \end{bmatrix} \sim \begin{bmatrix} 2 & -4 & 3 & -1 & 2 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 2 & -4 & 3 & -1 & 2 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Using variable names (x, y, z, s, t) for the solution we get:

$$\begin{aligned}
t &= \text{free} \\
s &= 0 \\
z &= -2t + s = -2t \\
y &= \text{free} \\
x &= (-2t + s - 3z + 4y)/2 = (-2t + 0 - 3(-2t) + 4y)/2 = (4t + 4y)/2 = 2t + 2y
\end{aligned}$$

The solution space is two-dimensional given by $(x, y, z, s, t) = (2t+2y, y, -2t, 0, t) = y(2, 1, 0, 0, 0)+ t(2, 0, -2, 0, 1)$. A basis for the solution space consists of $\{u_1, u_2\}$ where $u_1 = (2, 1, 0, 0, 0)$, and $u_2 = (2, 0, -2, 0, 1)$.

Solution to 4.99(b)

$$\begin{bmatrix} 1 & 2 & -1 & 3 & -4 \\ 2 & 4 & -2 & -1 & 5 \\ 2 & 4 & -2 & 4 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -1 & 3 & -4 \\ 0 & 0 & 0 & -1 & 5 \\ 0 & 0 & 0 & 4 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -1 & 3 & -4 \\ 0 & 0 & 0 & -1 & 5 \\ 0 & 0 & 0 & 0 & 18 \end{bmatrix}$$

Using variable names (x, y, z, s, t) for the solution we get:

$$\begin{aligned}
t &= 0 \\
s &= 0 \\
z &= \text{free} \\
y &= \text{free} \\
x &= -2y + z
\end{aligned}$$

The solution space is two-dimensional given by $(x, y, z, s, t) = (-2y+z, y, z, 0, 0) = y(-2, 1, 0, 0, 0)+ z(1, 0, 1, 0, 0)$. A basis for the solution space consists of $\{u_1, u_2\}$ where $u_1 = (-2, 1, 0, 0, 0)$, and $u_2 = (1, 0, 1, 0, 0)$.

Solution to 4.100(b)

$(x, y, z, s, t) = m_1(1, 1, 2, 1, 1) + m_2(1, 2, 1, 4, 3) + m_3(3, 5, 4, 9, 7)$

equivalent set of equations :

$$
\begin{aligned}
m_1 + m_2 + 3m_3 &= x \\
m_1 + 2m_2 + 5m_3 &= y \\
2m_1 + m_2 + 4m_3 &= z \\
m_1 + 4m_2 + 9m_3 &= s \\
m_1 + 3m_2 + 7m_3 &= t
\end{aligned}
$$

$$
\begin{bmatrix} 1 & 1 & 3 & x \\ 1 & 2 & 5 & y \\ 2 & 1 & 4 & z \\ 1 & 4 & 9 & s \\ 1 & 3 & 7 & t \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 3 & x \\ 0 & 1 & 2 & y-x \\ 0 & -1 & -2 & z-2x \\ 0 & 3 & 6 & s-x \\ 0 & 2 & 4 & t-x \end{bmatrix} \to
$$

$$
\begin{bmatrix} 1 & 1 & 3 & x \\ 0 & 1 & 2 & y-x \\ 0 & 0 & 0 & z-2x+y-x \\ 0 & 0 & 0 & s-x-3(y-x) \\ 0 & 0 & 0 & t-x-2(y-x) \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 3 & x \\ 0 & 1 & 2 & y-x \\ 0 & 0 & 0 & 3x+y+z \\ 0 & 0 & 0 & 2x-3y+s \\ 0 & 0 & 0 & x-2y+t \end{bmatrix} \to
$$

$$
\begin{aligned}
-3x + y + z &= 0 \\
2x - 3y + s &= 0 \\
x - 2y + t &= 0
\end{aligned}
$$

Solution to 1.53

(a)

The points $P(1, 2, 1, 2)$ and $Q(3, -5, 7, -9)$ are on the line, so a direction vector for the line is $v = \overrightarrow{PQ} = (2, -7, 6, -11)$. Using this direction vector and the point P on the line gives the parametric equations of the line:

$$
\frac{x-1}{2} = \frac{y-2}{-7} = \frac{z-1}{6} = \frac{t-2}{-11}
$$

These equations are equivalent to the vector form of the equation which is given in the answer in the text. The parametric equations are as given above. There is no difference in principle when working with equations of lines in R^n rather than R^3.

(b)

The line is perpendicular to the hyperplane given by the equation $2x + 4y + 6z - 8t = 5$, so the normal vector of the hyperplane $v = (2, 4, 6, -8)$ is the direction vector of the line. The line passes through the point $P(1, 1, 3, 3)$. Using this direction vector and the point P on the line gives the parametric equations of the line:

$$\frac{x-1}{2} = \frac{y-1}{4} = \frac{z-3}{6} = \frac{t-3}{-8}$$

These equations are equivalent to vector form of the equation which is given in the answer in the text. The parametric equations are as given above.

Solution to 1.56

(a)

The line is in the direction of $v = (4, -5, 7)$ and passes through the point $P(2, 5, -3)$. Therefore the parametric equations of the line are

$$\frac{x-2}{4} = \frac{y-5}{-5} = \frac{z+3}{7}$$

These equations are equivalent to the equation given in the text answer: $\mathbf{r} = (4t + 2, -5t + 5, 7t - 3)$, which is the vector form of the equation. The parametric equations are as given above.

(b)

The line is perpendicular to the plane, so the normal vector of the plane $v = (2, -3, 7)$ is the direction vector of the line. The line passes through the point $P(1, -5, 7)$. Therefore the parametric equations of the line are

$$\frac{x-1}{2} = \frac{y+5}{-3} = \frac{z-7}{7}$$

These equations are equivalent to the vector form of the equation given in the text answer. The parametric equations are as given above.

Solution to 3.52

(a)

for unique solution

$\det(\begin{bmatrix} 1 & -a \\ a & -4 \end{bmatrix}) \neq 0$ i.e. $-4 + a^2 \neq 0$ i.e. $a \neq 2, -2$

for more than one solution:

a=(2 or -2)

if a=2 :

$$\begin{aligned} x - 2y &= 1 \\ 2x - 4y &= b \end{aligned}$$

equivalent to

$$\begin{aligned} x - 2y &= 1 \\ x - 2y &= b/2 \end{aligned}$$

b/2=1, b=2

if a=-2:

$$\begin{aligned} x + 2y &= 1 \\ -2x - 4y &= b \end{aligned}$$

equivalent to

$$\begin{aligned} x + 2y &= 1 \\ x + 2y &= b/-2 \end{aligned}$$

-b/2=1, b=-2

(b)

for unique solution:

det($\begin{bmatrix} a & 3 \\ 12 & a \end{bmatrix}$) $\neq 0$ i.e. $a^2 - 36 \neq 0$ i.e. $a \neq 6, -6$
for more than one solution:

a=(6 or -6)

if a=6:

$$\begin{aligned} 6x + 3y &= 1 \\ 12x + 6y &= b \end{aligned}$$

equivalent to:

$$\begin{aligned} 6x + 3y &= 1 \\ 6x + 3y &= b/2 \end{aligned}$$

b/2=1, b=2

if a=-6:

$$\begin{aligned} -6x + 3y &= 1 \\ 12x - 6y &= b \end{aligned}$$

equivalent to:

$$\begin{aligned} -6x + 3y &= 1 \\ -6x + 3y &= -b/2 \end{aligned}$$

-b/2=1, b=-2

Solution to 3.53

(a)

The augmented matrix of the non-homogenous system is

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 2 & 3 & 6 & 10 \\ 3 & 6 & 10 & 14 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 1 & 2 & 2 \\ 0 & 3 & 4 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\begin{aligned} z &= 2 \\ y &= 2 - 2z = 2 - 2(2) = -2 \\ x &= 4 - 2z - y = 4 - 2(2) - (-2) = 2 \end{aligned}$$

The solution is unique: $(x, y, z) = (2, -2, 2)$.

(b)

The augmented matrix of the non-homogenous system is

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 2 & -3 & 8 & 7 \\ 3 & -4 & 13 & 8 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

The last equation has no solution, therefore the system is inconsistent.

(c)

The augmented matrix of the non-homogenous system is

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 2 & 3 & 8 & 4 \\ 5 & 8 & 19 & 11 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & 1 & -2 & 2 \\ 0 & 1 & -2 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned} z &= \text{free} \\ y &= 2 + 2z \\ x &= 3 - 3z - 2y = 3 - 3z - 2(2 + 2z) = -1 - 7z \end{aligned}$$

The general solution is $(x, y, z) = (-1 - 7z, 2 + 2z, z) = (-1, 2, 0) + z(-7, 2, 1)$.

Solution to 3.54

(a)

The augmented matrix of the non-homogenous system is

$$\left[\begin{array}{cc|c} 1 & -2 & 5 \\ 2 & 3 & 3 \\ 3 & 2 & 7 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -2 & 5 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -2 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned} y &= -1 \\ x &= 5 + 2y = 5 + (-1) = 3 \end{aligned}$$

The solution is unique: $(x, y) = (3, -1)$.

(b)

The augmented matrix of the non-homogenous system is

$$\left[\begin{array}{cccc|c} 1 & 2 & -3 & 2 & 2 \\ 2 & 5 & -8 & 6 & 5 \\ 3 & 4 & -5 & 2 & 4 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & -3 & 2 & 2 \\ 0 & 1 & -2 & 2 & 1 \\ 0 & 1 & -2 & 2 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & -3 & 2 & 2 \\ 0 & 1 & -2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned} t &= \text{free} \\ z &= \text{free} \\ y &= 1 - 2t + 2z \\ x &= 2 - 2t + 3z - 2y = 2 - 2t + 3z - 2(1 - 2t + 2z) = 2t - z \end{aligned}$$

The general solution is $(x, y, z, t) = (-z + 2t, 1 + 2z - 2t, z, t) = (0, 1, 0, 0) + z(-1, 2, 1, 0) + t(2, -2, 0, 1)$.

(c)

The augmented matrix of the non-homogenous system is

$$\left[\begin{array}{cccc|c} 1 & 2 & 4 & -5 & 3 \\ 3 & -1 & 5 & 2 & 4 \\ 5 & -4 & -6 & 9 & 2 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & 4 & -5 & 3 \\ 0 & 7 & 7 & -17 & 5 \\ 0 & 14 & 26 & -34 & 13 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & 4 & -5 & 3 \\ 0 & 7 & 7 & -17 & 5 \\ 0 & 0 & 12 & 0 & 3 \end{array}\right]$$

Note: You can see here that the question was intended to be inconsistent [as in the answer in the book], but there is a sign error typo - the '-6' in the third row of the coefficient matrix was obviously supposed to be a '+6'. As given the thing leads to a mess of fractions. Being all tough we solve that anyway...

$$\begin{aligned}
t &= \text{free} \\
12z &= 3 \longrightarrow z = 1/4 \\
7y &= 5 + 17t - 7z = 5 + 17t - 7/4 = 13/4 + 17t \longrightarrow y = 13/28 + 17t/7 \\
x &= 3 + 5t - 4z - 2y = 3 + 5t - 4(1/4) - 2(13/28 + 17t/7) = 15/14 + t/7
\end{aligned}$$

The general solution is $(x, y, z, t) = (15/14+t/7, 13/28+17t/7, 1/4, t) = (15/14, 13/28, 1/4, 0)+ t(1/7, 17/7, 0, 1)$.

Solution to 3.55

(a)

The augmented matrix of the non-homogenous system is

$$\left[\begin{array}{ccc|c} 2 & -1 & -4 & 2 \\ 4 & -2 & -6 & 5 \\ 6 & -3 & -8 & 8 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & -1 & -4 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 2 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & -1 & -4 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned}
2z &= 1 \longrightarrow z = 1/2 \\
y &= \text{free} \\
2x &= 2 + 4z + y = 2 + 4(1/2) + y = 4 + y \longrightarrow x = 2 + y/2
\end{aligned}$$

The general solution is $(x, y, z) = (2 + y/2, y, 1/2) = (2, 0, 1/2) + y(1/2, 1, 0)$.

(b)

The augmented matrix of the non-homogenous system is

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 3 \\ 2 & 4 & 4 & 3 & 9 \\ 3 & 6 & -1 & 8 & 10 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 3 \\ 0 & 0 & 2 & -1 & 1 \\ 0 & 0 & 2 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 3 \\ 0 & 0 & 2 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned}
t &= \text{free}\\
2z &= 1+t \longrightarrow z = 1/2 + t/2\\
y &= \text{free}\\
x &= 3-3t+z-2y = 3-3t+(1/2+t/2)-2y = 7/2-2y-5t/2
\end{aligned}$$

The general solution is $(x,y,z,t) = (7/2-2y-5t/2, y, 1/2+t/2, t) = (7/2,0,1/2,0) + y(-2,1,0,0) + t(-5/2,0,1/2,1)$.

Solution to 3.56

(a)

Let the coefficient matrix of the non-homogeneous system be A. The augmented matrix is

$$M = \left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 1 & -1 & a & 2 \\ 0 & a & 4 & b \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & 1 & a & 1 \\ 0 & a & 4 & b \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & 1 & a & 1 \\ 0 & 0 & 4-a^2 & b-a \end{array}\right]$$

For a unique solution we require $\mathrm{rank}(A) = 3$, so $4-a^2 \neq 0$, i.e. $a \neq \pm 2$. This condition is not affected by the value of b.

For more than one solution [and in particular not inconsistent] we require $\mathrm{rank}(A) = \mathrm{rank}(M) = 2$, so $4-a^2 = 0$ and $b-a = 0$. So $a = \pm 2$. There are therefore two cases which give more than one solution to the system: $(a,b) = (2,2)$ or $(-2,-2)$.

(b)

Let the coefficient matrix of the non-homogeneous system be A. The augmented matrix is

$$M = \left[\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 1 & a & 3 & 3 \\ 1 & 11 & a & b \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 0 & a-2 & 1 & 2 \\ 0 & 9 & a-2 & b-1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 0 & a-2 & 1 & 2 \\ 0 & 0 & C & D \end{array}\right]$$

where $C = 9-(a-2)^2 = -a^2+4a+5 = (-a+5)(a+1)$ and $D = 18-(b-1)(a-2) = 16-ab+a+2b$.

For a unique solution we require $\mathrm{rank}(A) = 3$, so $C \neq 0$, i.e. $a \neq 5, a \neq -1$. This condition is not affected by the value of b.

For more than one solution [and in particular not inconsistent] we require $\text{rank}(A) = \text{rank}(M) = 2$, so $C = D = 0$. Case 1: $C = 0$ with $a = 5$, and then $0 = D = 16 - 5b + 5 + 2b = 0$, so $b = 7$. Case 2: $C = 0$ with $a = -1$, and then $0 = D = 16 + b - 1 + 2b = 0$, so $b = -5$.There are therefore two cases which give more than one solution to the system: $(a, b) = (-1, -5)$ or $(5, 7)$.

(c)

Let the coefficient matrix of the non-homogeneous system be A. The augmented matrix is

$$M = \left[\begin{array}{ccc|c} 1 & 1 & a & 1 \\ 1 & a & 1 & 4 \\ a & 1 & 1 & b \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & a & 1 \\ 0 & a-1 & 1-a & 3 \\ 0 & a-1 & a^2-1 & a-b \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & a & 1 \\ 0 & a-1 & 1-a & 3 \\ 0 & 0 & a^2+a-2 & a-b-3 \end{array}\right]$$

For a unique solution we require $\text{rank}(A) = 3$, so $a^2 + a - 2 = (a+2)(a-1) \neq 0$, i.e. $a \neq -2, a \neq 1$. This condition is not affected by the value of b.

For more than one solution [and in particular not inconsistent] we require $\text{rank}(A) = \text{rank}(M) = 2$. So $(a+2)(a-1) = 0$ and $a - b - 3 = 0$. Case 1: $a = -2$, and then $0 = -2 - b - 3$, so $b = -5$. Case 2: $a = 1$, and then $0 = 1 - b - 3$, so $b = -2$. There are therefore two cases which give more than one solution to the system: $(a, b) = (-2, -5)$ or $(1, -2)$.