Linear Algebra, Spring 2005

Solutions

May 4, 2005

Solution to 1.51

(a) $\overrightarrow{PQ} = Q - P = (1, -6, -5) - (2, 3, -7) = (-1, -9, 2)$

(b) $\overrightarrow{PQ} = Q - P = (3, -5, 2, -4) - (1, -8, -4, 6) = (2, 3, 6, -10)$

Solution to 1.52

(a) The coefficients of the equation of the hyperplane H are the components of the normal vector. The given normal vector is $u = (2, 3, -5, 6)$, therefore the equation for hyperplane H will be of the form

$2x + 3y - 5z + 6t = k$.

As $P(1, 2, -3, 2)$ lies on the hyperplane, we substitute the value of P in the equation of the hyperplane and solve for k.

$k = 2(1) + 3(2) - 5(-3) + 6(2) = 35$

Thus, the equation of the hyperplane H is

$2x + 3y - 5z + 6t = 35$

Note: the variables x, y, z, t could have been chosen to be x_1, x_2, x_3, x_4 as in the book.

(b) The hyperplane H is parallel to the hyperplane given by the equation $2x_1 - 3x_2 + 5x_3 - 7x_4 = 4$, The two parallel hyperplanes will have identical normal vectors and their equations will be same except for a different constant value. The equation for the hyperplane H can be written as:

$2x_1 - 3x_2 + 5x_3 - 7x_4 = k$

Using the fact $P(3, -1, 2, 5)$ lies in the hyperplane, we solve for k.

$k = 2(3) - 3(-1) + 5(2) - 7(5) = -16$

Thus, the equation for the hyperplane H is

$2x_1 - 3x_2 + 5x_3 - 7x_4 = -16$

Solution to 1.55

(a) Since normal $\mathbf{N} = 3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$, the plane's equation will be in the form $3x - 4y + 5z = k$ where k is a constant to be determined.

Using the fact $P(1, 2, -3)$ lies in the plane, we solve for k.

$k = 3(1) - 4(2) + 5(-3) = -20$

The equation of the plane H is:

$3x - 4y + 5z = -20$

(b) Since the plane, H, is parallel to $4x + 3y - 2z = 11$, they should have identical normal vectors with a different constant. The plane H will have the equation

$4x + 3y - 2z = k$

where k is a constant to be determined.

Using the fact $Q(2, -1, 3)$ lies on the plane, we solve for k.

$k = 4(2) + 3(-1) - 2(3) = -1$

The equation of the plane H is:

$4x + 3y - 2z = -1$

(c) Extra part: Find the distance from Q to the plane H.

Ans. Using the formula on slide 34 (unit V), the distance from $Q(2, -1, 3)$ to the plane in (a) with equation $3x - 4y + 5z + 20 = 0$ is:

$$\begin{aligned} D &= \frac{|3(2) - 4(-1) + 5(3) + 20|}{\sqrt{3^2 + (-4)^2 + 5^2}} \\ &= |45|/\sqrt{50} = 9\sqrt{2}/2 \end{aligned}$$

The distance to the plane H in part (b) is zero, since Q lies on the plane.

Solution to 1.62

$u = (2, 1, 3), v = (4, -2, 2), w = (1, 1, 5)$

(a) $u \times v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 3 \\ 4 & -2 & 2 \end{vmatrix} = (2 + 6, -(4 - 12), -4 - 4) = (8, 8, -8)$

(b) $u \times w = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 3 \\ 1 & 1 & 5 \end{vmatrix} = (5 - 3, -(10 - 3), 2 - 1) = (2, -7, 1)$

(c) $v \times w = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -2 & 2 \\ 1 & 1 & 5 \end{vmatrix} = (-10 - 2, -(20 - 2), 4 + 2) = (-12, -18, 6)$

Note: There is a typo in the book for parts a and c.

Solution to 1.64

(a) $v \times w = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 1 & -1 & 2 \end{vmatrix} = (4 + 3, -(2 - 3), -1 - 2) = (7, 1, -3)$

The above vector is orthogonal to vectors v and w. To get a unit orthogonal vector u we need to normalize it.

$u = \frac{1}{\sqrt{(7)^2 + (1)^2 + (-3)^2}}(7, 1, -3) = \frac{1}{\sqrt{59}}(7, 1, -3)$

(b) $v \times w = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & 2 \\ 4 & -2 & -1 \end{vmatrix} = (1 + 4, -(-3 - 8), -6 + 4) = (5, 11, -2)$

The above vector is orthogonal to v and w, but it is not a unit vector. We need to normalize the vector to obtain u.

$u = \frac{1}{\sqrt{(5)^2 + (11)^2 + (-2)^2}}(5, 11, -2) = \frac{1}{\sqrt{150}}(5, 11, -2)$

Note: This can alternatively be written as $\frac{1}{\sqrt{150}}(5\mathbf{i} + 11\mathbf{j} - 2\mathbf{k})$

Solution to 7.64

A polynomial of degree ≤ 2 in the vector subspace W will take the form $f(t) = at^2 + bt + c$. By definition, if the inner product of $\langle f, h \rangle = 0$, then f and h are orthogonal. Hence:

$$\begin{aligned}
\langle f, h \rangle &= 0 \\
0 &= \int_0^1 f(t)h(t)dt \\
0 &= \int_0^1 (at^2 + bt + c)(2t + 1)dt \\
0 &= \int_0^1 \{2at^3 + 2bt^2 + 2ct + at^2 + bt + c\}dt \\
0 &= \int_0^1 2at^3 dt + \int_0^1 t^2(2b + a)dt + \int_0^1 t(2c + b)dt + \int_0^1 cdt \\
0 &= \left.\frac{2at^4}{4}\right|_0^1 + \left.\frac{t^3(2b + a)}{3}\right|_0^1 + \left.\frac{t^2(2c + b)}{2}\right|_0^1 + ct|_0^1 \\
0 &= \frac{a}{2} + \frac{(2b + a)}{3} + \frac{(2c + b)}{2} + c \\
0 &= \frac{5a}{6} + \frac{7b}{6} + 2c \\
0 &= 5a + 7b + 12c
\end{aligned}$$

Therefore, all the polynomials (in the form $at^2 + bt + c$) in the subspace W must conform to the above equation. As well, $\dim W = 2$.
let $a = 7, b = -5, c = 0$, then $f_1(t) = 7t^2 - 5t$
let $a = 12, b = 0, c = -5$, then $f_2(t) = 12t^2 - 5$
Therefore, the subspace W is spanned by $\{7t^2 - 5t, 12t^2 - 5\}$

Solution to 7.66

Let $w = (a, b, c, d, e)$ be a vector in the subspace W (which is orthogonal to u_1 and u_2). Then we require $\langle w, u_1 \rangle = 0$ and $\langle w, u_2 \rangle = 0$.

$$\langle w, u_1 \rangle = a + b + 3c + 4d + e \quad = \quad 0$$

$$\langle w, u_2\rangle = a + 2b + c + 2d + e = 0$$

Reducing the two equations gives:

$$\begin{aligned} a + b + 3c + 4d + e &= 0 \\ b - 2c - 2d &= 0 \end{aligned}$$

So $b = 2c + 2d$ and $a = -5c - 6d - e$ with free variables c, d, and e.
Solution is:

$$\begin{aligned} (a, b, c, d, e) &= (-5c - 6d - e, 2c + 2d, c, d, e) \\ &= c(-5, 2, 1, 0, 0) + d(-6, 2, 0, 1, 0) + e(-1, 0, 0, 0, 1) \end{aligned}$$

Therefore, $\{(-5, 2, 1, 0, 0), (-6, 2, 0, 1, 0), (-1, 0, 0, 0, 1)\}$ forms a basis for W.

Solution to 7.67

$w = (1, -2, -1, 3)$.
(a) To find the orthogonal basis, find a nonzero solution of $x - 2y - z + 3t = 0$.
And one such solution is $w_1 = (0, 0, 3, 1)$.

Now find a nonzero solution of the system: $x - 2y - z + 3t = 0, \quad 3z + t = 0$
and $w_2 = (0, 5, -1, 3)$ is one solution.

Next find a nonzero solution for the system: $x - 2y - z + 3t = 0, \quad 3z + t = 0, \quad 5y - z + 3t = 0$
$w_3 = (14, 2, 1, -3)$ is one solution.
$\{w_1, w_2, w_3\} = \{(0, 0, 3, 1), (0, 5, -1, 3), (14, 2, 1, -3)\}$ forms an orthogonal basis for $w^\perp$.

(b) The orthonormal basis is given by normalizing the basis vectors of the orthogonal basis.
$\left\{\frac{1}{\sqrt{10}}(0, 0, 3, 1), \frac{1}{\sqrt{35}}(0, 5, -1, 3), \frac{1}{\sqrt{210}}(14, 2, 1, -3)\right\}$

Solution to 7.68

(a) $u_1 = (1, 1, 2, 2)$, $u_2 = (0, 1, 2, -1)$ and let $w = (a, b, c, d)$

$\langle w, u_1 \rangle = 0 = a + b + 2c + 2d$

$\langle w, u_2 \rangle = 0 = b + 2c - d$

$b = -2c + d$ and $a = -3d$. The two free variables are c and d. Therefore,

$w = (-3d, -2c + d, c, d) = d(-3, 1, 0, 1) + c(0, -2, 1, 0)$

We require an <u>orthogonal</u> basis for W (i.e. $\{w_1, w_2\}$ a basis as well as $\langle w_1, w_2 \rangle = 0$). Start with $w_1 = (0, -2, 1, 0)$, then $w_2 = d(-3, 1, 0, 1) + c(0, -2, 1, 0)$. Therefore, we need to find c and d such that $\langle w_1, w_2 \rangle = 0$

$$\begin{aligned} \langle w_1, w_2 \rangle = \langle (0, -2, 1, 0), d(-3, 1, 0, 1) + c(0, -2, 1, 0) \rangle &= 0 \\ d(-2) + c(5) &= 0 \\ 5c &= 2d \end{aligned}$$

Pick $d = 5$ and $c = 2$, then $w_2 = 5(-3, 1, 0, 1) + 2(0, -2, 1, 0) = (-15, 1, 2, 5)$

Therefore, an orthogonal basis for W is $\{w_1, w_2\} = \{(0, -2, 1, 0), (-15, 1, 2, 5)\}$

Alternate Solution

Use Gram-Schmidt applied to $\{w_1, w_2\}$ any basis, say $w_1 = (-3, 1, 0, 1)$ and $w_2 = (0, -2, 1, 0)$

$v_1 = w_1 = (-3, 1, 0, 1)$

$$\begin{aligned} v_2 = w_2 - \mathrm{proj(w_2, v_1)} &= (0, -2, 1, 0) - \frac{\langle w_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 \\ v_2 &= (0, -2, 1, 0) + \frac{2}{11}(-3, 1, 0, 1) \\ 11v_2 &= 11(0, -2, 1, 0) + (-6, 2, 0, 2) \\ &= (-6, -20, 11, 2) \end{aligned}$$

Therefore $\{v_1, 11v_2\} = \{(-3, 1, 0, 1), (-6, -20, 11, 2)\}$ form an orthogonal basis.

(b) Normalize the basis from above [first choice]

$\{w_1, w_2\} = \{(0, -2, 1, 0), (-15, 1, 2, 5)\}$

$\|w_1\| = \sqrt{(-2)^2 + (1)^2} = \sqrt{5}$ and

$\|w_2\| = \sqrt{(-15)^2 + (1)^2 + (2)^2 + (5)^2} = \sqrt{255}$

Therefore, an orthonormal basis for W is $\{\frac{1}{\sqrt{5}}(0,-2,1,0), \frac{1}{\sqrt{255}}(-15,1,2,5)\}$

Solution to 7.69

(a) For the vectors to form an orthogonal set in R^4, each pair must be orthogonal:

$u_1 \cdot u_2 = (1)(1) + (1)(1) + (1)(-1) + (1)(-1) = 0$

$u_1 \cdot u_3 = (1)(1) + (1)(-1) + (1)(1) + (1)(-1) = 0$

$u_1 \cdot u_4 = (1)(1) + (1)(-1) + (1)(-1) + (1)(1) = 0$

$u_2 \cdot u_3 = (1)(1) + (1)(-1) + (-1)(1) + (-1)(-1) = 0$

$u_2 \cdot u_4 = (1)(1) + (1)(-1) + (-1)(-1) + (-1)(1) = 0$

$u_3 \cdot u_4 = (1)(1) + (-1)(-1) + (1)(-1) + (-1)(1) = 0$

Orthogonal sets are automatically linearly independent, so the four given vectors must be a basis.

(b) If $v = c_1u_1 + c_2u_2 + c_3u_3 + c_4u_4$ then $c_i = \dfrac{\langle v, u_i\rangle}{\langle u_i, u_i\rangle}$

$c_1 = \dfrac{\langle v, u_1\rangle}{\langle u_1, u_1\rangle} = \frac{1}{4}(1+3-5+6) = \frac{5}{4}$

$c_2 = \dfrac{\langle v, u_2\rangle}{\langle u_2, u_2\rangle} = \frac{1}{4}(1+3+5-6) = \frac{3}{4}$

$c_3 = \dfrac{\langle v, u_3\rangle}{\langle u_3, u_3\rangle} = \frac{1}{4}(1-3-5-6) = \frac{-13}{4}$

$c_4 = \dfrac{\langle v, u_4\rangle}{\langle u_4, u_4\rangle} = \frac{1}{4}(1-3+5+6) = \frac{9}{4}$

Therefore, $v = \frac{1}{4}[5, 3, -13, 9]_S$

(c) As above with $v = (a, b, c, d)$

$c_1 = \frac{1}{4}\langle v, u_1\rangle = \frac{1}{4}(a+b+c+d)$

$c_2 = \frac{1}{4}\langle v, u_2\rangle = \frac{1}{4}(a+b-c-d)$

$c_3 = \frac{1}{4}\langle v, u_3\rangle = \frac{1}{4}(a-b+c-d)$

$c_4 = \frac{1}{4}\langle v, u_4\rangle = \frac{1}{4}(a-b-c+d)$

Therefore, $v = \frac{1}{4}[a+b+c+d, a+b-c-d, a-b+c-d, a-b-c+d]_S$

(d) Note: $\|u_1\| = \|u_2\| = \|u_3\| = \|u_4\| = \sqrt{1^2+1^2+1^2+1^2} = \sqrt{4} = 2$

So each normalized vector would be $\hat{u}_i = (1/2)u_i$ where $i = 1, 2, 3, 4$.

Solution to 7.74

(b) Compute $\langle v, w\rangle = 1 - 6 + 7 + 8 = 10$ and $\|w\|^2 = (1 + 4 + 49 + 16) = 70$. Then

$$c = \frac{\langle v, w\rangle}{\langle w, w\rangle} = \frac{10}{70} = \frac{1}{7}$$

and

$$\text{proj}(v, w) = \langle v, \hat{w}\rangle\hat{w} = cw = \frac{1}{7}(1, -2, 7, 4)$$

(c)

$$\begin{aligned} c &= \frac{\langle v, w\rangle}{\langle w, w\rangle} \\ &= \frac{\int_0^1 t^3 dt + 3\int_0^1 t^2 dt}{\int_0^1 t^2 dt + 6\int_0^1 t dt + 9\int_0^1 dt} \\ &= \frac{\frac{t^4}{4}\Big|_0^1 + t^3\big|_0^1}{\frac{t^3}{3}\Big|_0^1 + 3t^2\big|_0^1 + 9t\big|_0^1} \\ &= \frac{\frac{1}{4} + 1}{\frac{1}{3} + 3 + 9} = \frac{\frac{5}{4}}{\frac{37}{3}} = \frac{15}{148} \end{aligned}$$

and

$$\text{proj}(v, w) = \langle v, \hat{w}\rangle\hat{w} = cw = \frac{15}{148}(t + 3)$$

Solution to 7.75

(a) Use the Gram-Schmidt algorithm:

$$w_1 = v_1 = (1, 1, 1, 1)$$

$$\begin{aligned} w_2 &= v_2 - \text{proj}(v_2, w_1) \\ &= v_2 - \langle v_2, \hat{w}_1\rangle\hat{w}_1 \\ &= v_2 - \frac{\langle v_2, w_1\rangle}{\langle w_1, w_1\rangle}w_1 \\ &= (1, -1, 2, 2) - \frac{4}{4}(1, 1, 1, 1) \\ &= (0, -2, 1, 1) \end{aligned}$$

$$
\begin{aligned}
w_3' &= v_3 - \operatorname{proj}(v_3, w_1) - \operatorname{proj}(v_3, w_2) \\
&= v_3 - \langle v_3, \hat{w}_1 \rangle \hat{w}_1 - \langle v_3, \hat{w}_2 \rangle \hat{w}_2 \\
&= v_3 - \frac{\langle v_3, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle v_3, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2 \\
&= (1, 2, -3, -4) - \frac{-4}{4}(1, 1, 1, 1) - \frac{-11}{6}(0, -2, 1, 1) \\
&= \left(\frac{12}{6}, \frac{-4}{6}, \frac{-1}{6}, \frac{-7}{6} \right)
\end{aligned}
$$

Choose $w_3 = 6w_3' = (12, -4, -1, -7)$ to clear fractions. $\{w_1, w_2, w_3\}$ form an orthogonal basis of U.

Normalize the orthogonal basis vectors $\{w_1, w_2, w_3\}$ to obtain an orthonormal basis $\{u_1, u_2, u_3\}$. We have $\|w_1\|^2 = 4$, $\|w_2\|^2 = 6$, and $\|w_3\|^2 = 210$. So the orthonormal basis is

$$
\begin{aligned}
u_1 &= \frac{1}{2}(1, 1, 1, 1) \\
u_2 &= \frac{1}{\sqrt{6}}(0, -2, 1, 1) \\
u_3 &= \frac{1}{\sqrt{210}}(12, -4, -1, -7)
\end{aligned}
$$

(b) The original vectors are not an orthogonal set, so we should use the orthogonal basis for U constructed in part (a): $\{w_1, w_2, w_3\}$ to calculate the projection of $v = (1, 2, -3, 4)$ onto U. First find the Fourier coefficients of v on each of the orthogonal vectors:

$$
\begin{aligned}
k_1 &= \frac{\langle v, w_1 \rangle}{\langle w_1, w_1 \rangle} = \frac{(1, 2, -3, 4) \cdot (1, 1, 1, 1)}{4} = 4/4 = 1 \\
k_2 &= \frac{\langle v, w_2 \rangle}{\langle w_2, w_2 \rangle} = \frac{(1, 2, -3, 4) \cdot (0, -2, 1, 1)}{6} = -3/6 = -1/2 \\
k_3 &= \frac{\langle v, w_3 \rangle}{\langle w_3, w_3 \rangle} = \frac{(1, 2, -3, 4) \cdot (12, -4, -1, -7)}{210} = -21/210 = -1/10
\end{aligned}
$$

Then

$$
\operatorname{proj}(v, U) = k_1 w_1 + k_2 w_2 + k_3 w_3 = 1(1, 1, 1, 1) - \frac{1}{2}(0, -2, 1, 1) - \frac{1}{10}(12, -4, -1, -7) = \frac{1}{5}(-1, 12, 3, 4)
$$

<u>Note:</u> There is typo in the text answer

Solution to 7.76

(a) Since $\{u_1, u_2\}$ is an orthogonal set, we first find the Fourier coefficients of v on each of the orthogonal vectors:

$$\begin{aligned} k_1 &= \frac{\langle v, u_1\rangle}{\langle u_1, u_1\rangle} = \frac{(1,2,3,4,6)\cdot(1,2,1,2,1)}{(1,2,1,2,1)\cdot(1,2,1,2,1)} = \frac{22}{11} = 2 \\ k_2 &= \frac{\langle v, u_2\rangle}{\langle u_2, u_2\rangle} = \frac{(1,2,3,4,6)\cdot(1,-1,2,-1,1)}{(1,-1,2,-1,1)\cdot(1,-1,2,-1,1)} = \frac{7}{8} \end{aligned}$$

The projection of $v = (1,2,3,4,6)$ onto W is then given by constructing the projections onto each of the components u_1 and u_2:

$$w = \text{proj}(v, W) = k_1u_1 + k_2u_2 = 2(1,2,1,2,1) + \frac{7}{8}(1,-1,2,-1,1) = \frac{1}{8}(23,25,30,25,23)$$

<u>Note:</u> There is typo in the text answer.

(b) $\{v_1, v_2\}$ is not an orthogonal set, so first we need to find an orthogonal basis for W. Apply the Gram-Schmidt algorithm to $\{v_1, v_2\}$.

$$\begin{aligned} w_1 &= v_1 = (1,2,1,2,1) \\ w_2 &= v_2 - \frac{\langle v_2, w_1\rangle}{\langle w_1, w_1\rangle}w_1 = (1,0,1,5,-1) - \frac{11}{11}(1,2,1,2,1) = (0,-2,0,3,-2) \end{aligned}$$

Now we can calculate the projection of $v = (1,2,3,4,6)$ onto W by constructing the projections onto each of the orthogonal vectors w_1 and w_2.
Find the Fourier coefficients of v on each of the orthogonal vectors:

$$\begin{aligned} k_1 &= \frac{\langle v, w_1\rangle}{\langle w_1, w_1\rangle} = \frac{(1,2,3,4,6)\cdot(1,2,1,2,1)}{(1,2,1,2,1)\cdot(1,2,1,2,1)} = \frac{22}{11} = 2 \\ k_2 &= \frac{\langle v, w_2\rangle}{\langle w_2, w_2\rangle} = \frac{(1,2,3,4,6)\cdot(0,-2,0,3,-2)}{(0,-2,0,3,-2)\cdot(0,-2,0,3,-2)} = \frac{-4}{17} \end{aligned}$$

Then

$$\text{proj}(v, W) = k_1w_1 + k_2w_2 = 2(1,2,1,2,1) - \frac{4}{17}(0,-2,0,3,-2) = \frac{1}{17}(34,76,34,56,42)$$

Solution to 7.77

Following the hint - which gives you an orthogonal basis for P_2 obtained in problem 7.22 - we find the Fourier coefficients of $f(t) = t^3$ with respect to $w_1 = 1$, $w_2 = 2t - 1$ and $w_3 = 6t^2 - 6t + 1$.

$$k_1 = \frac{\langle f, w_1\rangle}{\langle w_1, w_1\rangle} = \frac{\int_0^1 t^3 dt}{\int_0^1 dt} = \frac{\frac{t^4}{4}\Big|_0^1}{t|_0^1} = \frac{1}{4}$$

$$\begin{aligned} k_2 &= \frac{\langle f, w_2\rangle}{\langle w_2, w_2\rangle} = \frac{\int_0^1 t^3(2t-1)dt}{\int_0^1 (2t-1)^2 dt} = \frac{\int_0^1 (2t^4 - t^3)dt}{\int_0^1 (4t^2 - 4t + 1)dt} \\ &= \frac{\left(\frac{2t^5}{5} - \frac{t^4}{4}\right)\Big|_0^1}{\left(\frac{4t^3}{3} - 2t^2 + t\right)\Big|_0^1} = \frac{\frac{3}{20}}{\frac{1}{3}} = \frac{9}{20} \end{aligned}$$

$$\begin{aligned} k_3 &= \frac{\langle f, w_3\rangle}{\langle w_3, w_3\rangle} = \frac{\int_0^1 t^3(6t^2 - 6t + 1)dt}{\int_0^1 (6t^2 - 6t + 1)^2 dt} = \frac{\int_0^1 (6t^5 - 6t^4 + t^3)dt}{\int_0^1 (36t^4 - 72t^3 + 48t^2 - 12t + 1)dt} \\ &= \frac{\left(t^6 - \frac{6t^5}{5} + \frac{t^4}{4}\right)\Big|_0^1}{\left(\frac{36t^5}{5} - 18t^4 + 16t^3 - 6t^2 + t\right)\Big|_0^1} = \frac{\frac{1}{20}}{\frac{1}{5}} = \frac{1}{4} \end{aligned}$$

Therefore, $\text{proj}(f, W) = \frac{1}{4}(1) + \frac{9}{20}(2t-1) + \frac{1}{4}(6t^2 - 6t + 1) = \frac{3}{2}t^2 - \frac{3}{5}t + \frac{1}{20}$

Solution to 7.94

Apply the Gram-Schmidt algorithm. Calculate:

$$w_1 = u_1 = (1, i, 1)$$

$$\begin{aligned} w_2' &= u_2 - \text{proj}(u_2, w_1) \\ &= u_2 - \langle u_2, \hat{w}_1\rangle \hat{w}_1 \\ &= u_2 - \frac{\langle u_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1 \end{aligned}$$

$$
\begin{aligned}
&= (1+i,0,2) - \left[\frac{(1+i,0,2)\cdot(1,i,1)}{(1,i,1)\cdot(1,i,1)}\right](1,i,1) \\
&= (1+i,0,2) - \frac{3+i}{3}(1,i,1) \\
w_2 &= 3w_2' \\
&= (3+3i,0,6) - (3+i,-1+3i,3+i) \\
&= (2i,1-3i,3-i)
\end{aligned}
$$

The vectors $\{w_1, w_2\}$ form an orthogonal basis for the space spanned by the given vectors. Normalize vectors w_1 and w_2 to obtain an orthonormal basis $\{\hat{w}_1, \hat{w}_2\}$.We have

$$
\begin{aligned}
\|w_1\|^2 &= 1+|i|^2+1 = 3 \\
\|w_2\|^2 &= |2i|^2+|1-3i|^2+|3-i|^2 = 4+(1+9)+(9+1) = 24
\end{aligned}
$$

So the orthonormal basis $\{\hat{w}_1, \hat{w}_2\} = \left\{\frac{1}{\sqrt{3}}(1,i,1), \frac{1}{\sqrt{24}}(2i,1-3i,3-i)\right\}$

Solution to 7.81

Practically speaking the difficult part in constructing an orthogonal matrix is getting mutually orthogonal rows. Once you have that matrix it is simple enough to normalize each row. We'll proceed to get the orthogonality first, in which case the fractions ($\frac{1}{3}$ etc) are an unnecessary nuisance so we'll drop them.

The desired matrix P is supposed to be symmetric, so it is in the form

$$
P = \begin{bmatrix} 1 & 2 & 2 \\ 2 & a & b \\ 2 & b & c \end{bmatrix}
$$

where the a, b, c unknowns have to be found. For mutually orthogonal rows we require:

$$
\begin{aligned}
2+2a+2b &= 0 \\
2+2b+2c &= 0 \\
4+ab+bc &= 0
\end{aligned}
$$

The first two equations are linear [non-homogeneous] in the three unknowns a, b, c. The augmented matrix of the system is already in reduced form [the constants are put on the right hand side of the equations of course]:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ 0 & 1 & 1 & -1 \end{array}\right]$$

Using back-substitution we get $b = -c - 1$ and $a = -b - 1 = -(-c-1) - 1 = c$. So the general solution is $(a, b, c) = (c, -c-1, c)$ any c.
Now we can substitute these values into the third [non-linear] orthogonality equation above: $0 = 4 + ab + bc = 4 + c(-c-1) + (-c-1)c = c^2 + c - 2 = (c-1)(c+2) = 0$. So we have two possible solutions: $c = 1$ and $c = -2$. These choices correspond to values of $(a, b, c) = (1, -2, 1)$ and $(-2, 1, -2)$. The possible choices for the [non-normalized] P matrix are therefore:

$$P_1' = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} \quad P_2' = \begin{bmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{bmatrix}$$

To complete the problem and construct orthogonal matrices we have to normalize the rows of P_1' and P_2'. Therefore we have

$$P_1 = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} \quad P_2 = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{bmatrix}$$

Solution to 9.56b

From the characteristic equation

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 4-\lambda & -1 \\ -1 & 4-\lambda \end{vmatrix} \\ &= (4-\lambda)^2 - (-1)^2 = 16 - 8\lambda + \lambda^2 - 1 = \lambda^2 - 8\lambda + 15 = (\lambda-3)(\lambda-5) \end{aligned}$$

Accordingly, $\lambda = 3$ and $\lambda = 5$ are eigenvalues of A.

Subtract $\lambda = 3$ on the diagonal of A, we have

$$A - 3I = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \text{ corresponding to } \begin{cases} x - y = 0 \\ -x + y = 0 \end{cases} \quad \text{or } x - y = 0$$

One free variable, and $v_1 = (1, 1)$ is a nonzero solution.

Subtract $\lambda = 5$ down the diagonal of A, we have

$$A - 5I = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}, \text{ corresponding to } \begin{cases} -x - y = 0 \\ -x - y = 0 \end{cases} \quad \text{or } x + y = 0$$

One free variable, and $v_2 = (1, -1)$ is a nonzero solution.

Since A is symmetric, the eigenvectors v_1 and v_2 are orthogonal.

Normalize v_1 and v_2 to get the unit vectors:

$\hat{v}_1 = (1/\sqrt{2}, 1/\sqrt{2})$ and $\hat{v}_2 = (1/\sqrt{2}, -1\sqrt{2})$

P is the orthogonal matrix whose columns are the unit vectors $\hat{v}_1$ and $\hat{v}_2$ respectively:

$$P = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \quad \text{and} \quad D = P^{-1}AP = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$$

Solution to 9.56c

From the characteristic equation

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 7-\lambda & 3 \\ 3 & -1-\lambda \end{vmatrix} \\ &= (7-\lambda)(-1-\lambda) - (3)^2 = -7 - 6\lambda + \lambda^2 - 9 = \lambda^2 - 6\lambda - 16 = (\lambda - 8)(\lambda + 2) \end{aligned}$$

Accordingly, $\lambda = 8$ and $\lambda = -2$ are eigenvalues of A.

Subtract $\lambda = 8$ on the diagonal of A, we have

$$A - 8I = \begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix}, \text{ corresponding to } \begin{cases} -x + 3y = 0 \\ 3x - 9y = 0 \end{cases} \quad \text{or } x - 3y = 0$$

One free variable, and $v_1 = (3, 1)$ is a nonzero solution.

Subtract $\lambda = -2$ down the diagonal of A, we have

$A - (-2)I = \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}$, corresponding to $\begin{cases} 9x + 3y = 0 \\ 3x + y = 0 \end{cases}$ or $3x + y = 0$

One free variable, and $v_2 = (1, -3)$ is a nonzero solution.

Since A is symmetric, the eigenvectors v_1 and v_2 are orthogonal.

Normalize v_1 and v_2 to get the unit vectors:

$\hat{v}_1 = (3/\sqrt{10}, 1/\sqrt{10})$ and $\hat{v}_2 = (1/\sqrt{10}, -3\sqrt{10})$

P is the orthogonal matrix whose columns are the unit vectors $\hat{v}_1$ and $\hat{v}_2$ respectively:

$P = \begin{bmatrix} 3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & -3/\sqrt{10} \end{bmatrix}$ and $D = P^{-1}AP = \begin{bmatrix} 8 & 0 \\ 0 & -2 \end{bmatrix}$

Solution to 9.57a

From the characteristic equation:

$$\begin{aligned} \det(B - \lambda I) &= \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} \xrightarrow{R_2 - R_1} \begin{vmatrix} -\lambda & 1 & 1 \\ 1+\lambda & -\lambda - 1 & 0 \\ 1 & 1 & -\lambda \end{vmatrix} \\ &= (1+\lambda) \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & -\lambda \end{vmatrix} \xrightarrow{C_1 + C_2} (1+\lambda) \begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & -1 & 0 \\ 2 & 1 & -\lambda \end{vmatrix} \\ &= -(1+\lambda) \begin{vmatrix} 1-\lambda & 1 \\ 2 & -\lambda \end{vmatrix} = -(1+\lambda)(\lambda^2 - \lambda - 2) = (1+\lambda)^2(2-\lambda) \end{aligned}$$

Accordingly, the eigenvalues of B are $\lambda = -1$ (multiplicity 2) and $\lambda = 2$ (multiplicity 1).

Find an orthogonal basis of each eigenspace. Subtract $\lambda = -1$ down the diagonal of B:

$B - (-1)I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, corresponding to $\begin{cases} x + y + z = 0 \\ x + y + z = 0 \\ x + y + z = 0 \end{cases}$ or $x + y + z = 0$

The system has two independent solutions. One solution is $v_1 = (1, -1, 0)$.

Let the second solution be $v_2 = (a, b, c)$, which is orthogonal to v_1. It should satisfy the following two conditions:

$$\begin{cases} a + b + c &= 0 \\ a - b &= 0 \end{cases}$$

One such solution is $v_2 = (1, 1, -2)$

For the eigenvalue $\lambda = 2$, subtract $\lambda = 2$ down the diagonal of B:

$$B - 2I = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}, \text{ corresponding to } \begin{cases} -2x + y + z = 0 \\ x - 2y + z = 0 \\ x + y - 2z = 0 \end{cases} \text{ or } \begin{cases} x - y = 0 \\ x - z = 0 \end{cases}$$

One free variable and $v_3 = (1, 1, 1)$ is a nonzero solution.

Since B is symmetric, the eigenvector v_3 is orthogonal to eigenvectors v_1 and v_2.

The maximal orthogonal set of eigenvectors $S = \{v_1, v_2, v_3\}$

Normalize v_1, v_2 and v_3 to get the orthonormal basis:

$$\begin{aligned} \hat{v}_1 &= v_1/\sqrt{2} = (1/\sqrt{2}, -1/\sqrt{2}, 0) \\ \hat{v}_2 &= v_2/\sqrt{6} = (1/\sqrt{6}, 1/\sqrt{6}, -2/\sqrt{6}) \\ \hat{v}_3 &= v_3/\sqrt{3} = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}) \end{aligned}$$

P is the orthogonal matrix whose columns are $\hat{v}_1, \hat{v}_2, \hat{v}_3$:

$$P = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \quad \text{and} \quad D = P^{-1}BP = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$