Linear Algebra, Spring 2005

Solutions

May 4, 2005

Question 7.57

We examine the 3 axioms in the definition.

1. For I_1 (Linear Property):
   $f(au + bu', v) = (ax_1 + bx'_1)y_1 - 2(ax_1 + bx'_1)y_2 - 2(ax_2 + bx'_2)y_1 + 5(ax_2 + bx'_2)y_2$
   $= ax_1y_1 - 2ax_1y_2 - 2ax_2y_1 + 5ax_2y_2 + bx'_1y_1 - 2bx'_1y_2 - 2bx'_2y_1 + 5bx'_2y_2$
   and
   $af(u, v) + bf(u', v) = ax_1y_1 - 2ax_1y_2 - 2ax_2y_1 + 5ax_2y_2 + bx'_1y_1 - 2bx'_1y_2 - 2bx'_2y_1 + 5bx'_2y_2$
   hence, $f(au + bu', v) = af(u, v) + bf(u', v)$

2. For I_2 (Symmetric Property):
   $f(u, v) = x_1y_1 - 2x_1y_2 - 2x_2y_1 + 5x_2y_2$ and
   $f(v, u) = y_1x_1 - 2y_1x_2 - 2y_2x_1 + 5y_2x_2$
   hence, $f(u, v) = f(v, u)$

3. For I_3 (Positive Definite Property):
   $f(u, u) = x_1x_1 - 2x_1x_2 - 2x_2x_1 + 5x_2x_2 = x_1^2 - 4x_1x_2 + 4x_2^2 + x_2^2 = (x_1 - 2x_2)^2 + x_2^2$
   hence $f(u, u) \geq 0$, and $f(u, u) = 0$ only if $x_1, x_2 = 0$ (i.e. $u = 0$).
   Note: In checking a formula for a proposed inner product, the positive definite axiom often involves factorizing into a sum of squares as in this example.

Question 7.59

(a) $\langle u, v\rangle = 1 \times 2 + (-3) \times 5 = -13$

(b) $\langle u, v\rangle = 1 \times 2 - 2 \times 1 \times 5 - 2 \times (-3) \times 2 + 5 \times (-3) \times 5 = -71$

(c) $\|v\| = \sqrt{2 \times 2 + 2 \times 5} = \sqrt{29}$

(d) $\|v\| = \sqrt{2 \times 2 - 2 \times 2 \times 5 - 2 \times 5 \times 2 + 5 \times 5 \times 5} = \sqrt{89}$

Question 7.60(b)

We can simply check I_3: $\langle u, u\rangle = x_1x_2x_3 + x_1x_2x_3 = 2x_1x_2x_3$, which cannot guarantee to be greater or equal than 0

Question 7.62

The C-S inequality is proved in problem 7.8 (real case) and problem 7.50 (complex case). The proof given in the lecture is almost the same as the one in 7.50. The real case is a special case of that of course. In the proof we evaluated: $\|u - \langle u, \hat{v}\rangle\hat{v}\|^2 \geq 0$. Equality in the C-S inequality occurs when that norm is zero, i.e. when the vector is the zero vector. This is when:

$$u = \langle u, \hat{v}\rangle\hat{v} = [\langle u, v\rangle/\|v\|^2]v$$

Therefore u and v are linear dependent. $u = Cv$, where C is a complex scalar having the value of $\langle u, v\rangle/\|v\|^2$

Question 1.41

$u = (1, -2, 4), v = (3, 5, 1), w = (2, 1, -3)$

(a)

$$\begin{aligned} 3u - 2v &= 3(1,-2,4) - 2(3,5,1) \\ &= (-3,-16,2) \end{aligned}$$

(b)

$$\begin{aligned} 5u + 3v - 4w &= 5(1,-2,4) + 3(3,5,1) - 4(2,1,-3) \\ &= (5+9-8, -10+15-4, 20+3+12) \\ &= (6,1,35) \end{aligned}$$

(c)

$$\begin{aligned} u \cdot v &= (1,-2,4)\cdot(3,5,1) = (1)(3) + (-2)(5) + (4)(1) = -3 \\ u \cdot w &= (1,-2,4)\cdot(2,1,-3) = (1)(2) + (-2)(1) + (4)(-3) = -12 \\ v \cdot w &= (3,5,1)\cdot(2,1,-3) = (3)(2) + (5)(1) + (1)(-3) = 8 \end{aligned}$$

Note: error in book for $u \cdot w$

(d)

$$\begin{aligned} \|u\| &= \sqrt{u \cdot u} = \sqrt{(1)^2 + (-2)^2 + (4)^2} = \sqrt{21} \\ \|v\| &= \sqrt{v \cdot v} = \sqrt{(3)^2 + (5)^2 + (1)^2} = \sqrt{35} \\ \|w\| &= \sqrt{w \cdot w} = \sqrt{(2)^2 + (1)^2 + (-3)^2} = \sqrt{14} \end{aligned}$$

(e)

$$\cos\theta = \frac{u \cdot v}{\|u\|\|v\|} = \frac{-3}{\sqrt{21}\sqrt{35}}$$

Using the results from section c and d.

(f)

$$d(u,v) = \|u-v\| = \sqrt{(2)^2+(7)^2+(-3)^2} = \sqrt{62}$$

(g)

$$\text{proj}(u,v) = (u\cdot\hat{v})\hat{v} = \frac{u\cdot v}{\|v\|^2}v = \frac{-3}{35}v = \frac{-3}{35}(3,5,1)$$

using the results from section c and d

Question 1.43

$u = (2,-5,4,6,-3), v = (5,-2,1,-7,-4)$

(a)

$$\begin{aligned} 4u-3v &= 4(2,-5,4,6,-3)-3(5,-2,1,-7,-4) \\ &= (-7,-14,13,45,0) \end{aligned}$$

Note: there is a typo in the text book for this question

(b)

$$\begin{aligned} 5u+2v &= 5(2,-5,4,6,-3)+2(5,-2,1,-7,-4) \\ &= (20,-29,22,16,-23) \end{aligned}$$

(c)

$$\begin{aligned} u\cdot v &= (2,-5,4,6,-3)\cdot(5,-2,1,-7,-4) \\ &= (2)(5)+(-5)(-2)+(4)(1)+(6)(-7)+(-3)(-4) = -6 \end{aligned}$$

(d)

$$\begin{aligned}\|u\| &= \sqrt{u\cdot u} = \sqrt{(2)^2+(-5)^2+(4)^2+(6)^2+(-3)^2} = \sqrt{90}\\ \|v\| &= \sqrt{v\cdot v} = \sqrt{(5)^2+(-2)^2+(1)^2+(-7)^2+(-4)^2} = \sqrt{95}\end{aligned}$$

(e)

$$\text{proj}(u,v) = (u\cdot\hat{v})\hat{v} = \frac{u\cdot v}{\|v\|^2} = \frac{-6}{95}v = \frac{-6}{95}(5,-2,1,-7,-4)$$

using the results from section c and d

(f)

$$d(u,v) = \|u-v\| = \sqrt{(3)^2+(3)^2+(-3)^2+(-13)^2+(-1)^2} = \sqrt{197}$$

Question 1.44c

$$\begin{aligned}\|w\| &= \sqrt{w\cdot w} = \sqrt{(1/2)^2+(-1/3)^2+(3/4)^2} = \sqrt{133/144} = \sqrt{133}/12\\ \hat{w} &= \frac{1}{\|w\|}w = \frac{12}{\sqrt{133}}(1/2,-1/3,3/4) = \frac{1}{\sqrt{133}}(6,-4,9)\end{aligned}$$

Question 1.45

(a)

$$\begin{aligned}\|u\| &= \sqrt{u\cdot u} = \sqrt{(1)^2+(2)^2+(-2)^2} = \sqrt{9} = 3\\ \|v\| &= \sqrt{v\cdot v} = \sqrt{(3)^2+(-12)^2+(4)^2} = \sqrt{169} = 13\\ \|u+v\| &= \sqrt{(u+v)\cdot(u+v)} = \sqrt{(1+3)^2+(2-12)^2+(-2+4)^2} = \sqrt{(4)^2+(-10)^2+(2)^2} = \sqrt{120}\\ \|ku\| &= \sqrt{(ku)\cdot(ku)} = \sqrt{(3)^2+(6)^2+(6)^2} = \sqrt{81} = 9\end{aligned}$$

(b)

Prove $\|ku\| = |k|\|u\|$

$RS = |k|\|u\| = |-3| \cdot \|u\| = 3(3) = 9 = LS$

Using results from part (a)

Prove $\|u + v\| \leq \|u\| + \|v\|$

$RS = \|u\| + \|v\| = 3 + 13 = 16 > \sqrt{120}$

Therefore, LS < RS

Using the results form part (a)

Question 7.91

(a)

$$\begin{aligned} \langle u, v \rangle &= (1+i)(\overline{3-4i}) + (3)(\overline{1+i}) + (4-i)(\overline{2i}) \\ &= (1+i)(3+4i) + (3)(1-i) + (4-i)(2i) \\ &= -4i \end{aligned}$$

(b)

$$\langle v, u \rangle = \overline{\langle u, v \rangle} = \overline{-4i} = 4i$$

(c)

$$\|u\| = \sqrt{\langle u, u \rangle} = \sqrt{1+1+9+16+1} = \sqrt{28} = 2\sqrt{7}$$

(d)

$$\|v\| = \sqrt{\langle v, v \rangle} = \sqrt{9+16+1+1+4} = \sqrt{31}$$

(e)

$$\begin{aligned} d(u,v) &= \|u-v\| = \|(1+i,3,4-i)-(3-4i,1+i,2i)\| \\ &= \|(5i-2,2-i,4-3i)\| = \sqrt{4+25+4+1+16+9} \\ &= \sqrt{59} \end{aligned}$$

Question 7.92

(a)

$$\begin{aligned} \langle v,w\rangle &= (3+i)(\overline{5+i})+(5-2i)(\overline{1+i}) \\ &= (3+i)(5-i)+(5-2i)(1-i) \\ &= 19-5i \end{aligned}$$

$$\begin{aligned} \langle w,w\rangle &= 25+1+1+1=28 \\ k &= \frac{\langle v,w\rangle}{\langle w,w\rangle} \\ &= \frac{1}{28}(19-5i) \end{aligned}$$

(b)

$$\begin{aligned} \langle v,w\rangle &= (1-i)(1)+(3i)(\overline{2-i})+(1+i)(\overline{3+2i}) \\ &= (1-i)+(3i)(2+i)+(1+i)(3-2i) \\ &= 3+6i \end{aligned}$$

$$\begin{aligned} \langle w,w\rangle &= 1+4+1+9+4=19 \\ k &= \frac{\langle v,w\rangle}{\langle w,w\rangle} \\ &= \frac{1}{19}(3+6i) \end{aligned}$$

Question 7.93

We examine the 3 axioms in the definition.

1. For I_1^* (Linear Property):

   $f(au+bu',v) = (az_1+bz_1')\bar{w}_1+(1+i)(az_1+bz_1')\bar{w}_2+(1-i)(az_2+bz_2')\bar{w}_1+3(az_2+bz_2')\bar{w}_2$

   and

   $$\begin{aligned} af(u,v)+bf(u',v) &= az_1\bar{w}_1+(1+i)az_1\bar{w}_2+(1-i)az_2\bar{w}_1+3az_2\bar{w}_2 \\ &\quad +bz_1'\bar{w}_1+(1+i)bz_1'\bar{w}_2+(1-i)bz_2'\bar{w}_1+3bz_2'\bar{w}_2 \\ &= (az_1+bz_1')\bar{w}_1+(1+i)(az_1+bz_1')\bar{w}_2 \\ &\quad +(1-i)(az_2+bz_2')\bar{w}_1+3(az_2+bz_2')\bar{w}_2 \end{aligned}$$

   hence, $f(au+bu',v) = af(u,v)+bf(u',v)$

2. For I_2^* (Conjugate Symmetric Property):

   $f(u,v) = z_1\bar{w}_1+(1+i)z_1\bar{w}_2+(1-i)z_2\bar{w}_1+3z_2\bar{w}_2$

   and

   $f(v,u) = w_1\bar{z}_1+(1+i)w_1\bar{z}_2+(1-i)w_2\bar{z}_1+3w_2\bar{z}_2$

   $f\overline{(v,u)} = \bar{w}_1z_1+\overline{(1+i)}\bar{w}_1z_2+\overline{(1-i)}\bar{w}_2z_1+3\bar{w}_2z_2$

   $f\overline{(v,u)} = z_1\bar{w}_1+(1+i)z_1\bar{w}_2+(1-i)z_2\bar{w}_1+3z_2\bar{w}_2$

   hence, $f(u,v) = f\overline{(v,u)}$

3. For I_3^* (Positive Definite Property):

   $$\begin{aligned} f(u,u) &= z_1\bar{z}_1+(1+i)z_1\bar{z}_2+(1-i)z_2\bar{z}_1+3z_2\bar{z}_2 \\ &= [z_1\bar{z}_1+(1+i)z_1\bar{z}_2+(1-i)z_2\bar{z}_1+2z_2\bar{z}_2]+[z_2\bar{z}_2] \\ &= (\bar{z}_1+(1+i)\bar{z}_2)(z_1+(1-i)z_2)+z_2\bar{z}_2 \\ &= (\bar{z}_1+\overline{(1-i)}\bar{z}_2)(z_1+(1-i)z_2)+z_2\bar{z}_2 \\ &= \overline{z_1+(1-i)z_2)}(z_1+(1-i)z_2)+z_2\bar{z}_2 \\ &= |z_1+(1-i)z_2)|^2+|z_2|^2 \\ &\geq 0 \end{aligned}$$

and equals zero only when $z_1 = z_2 = 0$, i.e. when $u = (0, 0)$.