# Linear Algebra, Spring 2005

Solutions

May 4, 2005

## Question 9.45

### (a)

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -3 \\ 2 & -5-\lambda \end{vmatrix} = \lambda^2 + 3\lambda - 4 = (\lambda + 4)(\lambda - 1)$$

The roots $\lambda = -4$ and $\lambda = 1$ are the eigenvalues of $A$

Subtract $\lambda = -4$ down the diagonal of $A$, we have

$$M = A - (-4)I = \begin{bmatrix} 6 & -3 \\ 2 & -1 \end{bmatrix}, \text{ corresponding to } \begin{cases} 6x - 3y = 0 \\ 2x - y = 0 \end{cases}$$

The system has only one free variable, and $v_1 = (1, 2)$ is a nonzero solution. $v_1$ represents an eigenvector belonging to and spanning the eigenspace of $\lambda = -4$

Subtract $\lambda = 1$ down the diagonal of $A$, we have

$$M = A - I = \begin{bmatrix} 1 & -3 \\ 2 & -6 \end{bmatrix}, \text{ corresponding to } \begin{cases} x - 3y = 0 \\ 2x - 6y = 0 \end{cases}$$

The system has only one free variable, and $v_2 = (3, 1)$ is a nonzero solution.

$v_2$ represents an eigenvector belonging to and spanning the eigenspace of $\lambda = 1$.

A nonsingular matrix $P$ is a matrix whose columns are $v_1$ and $v_2$.

$$P = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$$

The diagonal matrix $D = P^{-1}AP = \begin{bmatrix} -4 & 0 \\ 0 & 1 \end{bmatrix}$

The diagonal entries of $D$ are the eigenvalues of $A$ corresponding to the eigenvectors appearing in $P$

### (b)

$\det(B - \lambda I) = \begin{vmatrix} 2-\lambda & 4 \\ -1 & 6-\lambda \end{vmatrix} = \lambda^2 - 8\lambda + 16 = (\lambda - 4)^2$

The equation has two equal roots, $\lambda = 4$. These are the eigenvalues of $A$

Subtract $\lambda = 4$ down the diagonal of $B$, we have

$M = B - 4I = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}$, corresponding to $\begin{cases} -2x + 4y = 0 \\ -x + 2y = 0 \end{cases}$

The system has only one free variable, and $v = (2, 1)$ is a nonzero solution There is only one independent eigenvector, so $B$ can not be diagonalized.

## Question 9.46

### (a)

$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -1 \\ -2 & 3-\lambda \end{vmatrix} = \lambda^2 - 5\lambda + 4 = (\lambda - 4)(\lambda - 1)$

The roots $\lambda = 4$ and $\lambda = 1$ are the eigenvalues of $A$

Subtract $\lambda = 4$ down the diagonal of $A$, we have

$M = A - (4)I = \begin{bmatrix} -2 & -1 \\ -2 & -1 \end{bmatrix}$, corresponding to $\begin{cases} -2x - y = 0 \\ -2x - y = 0 \end{cases}$

The system has only one free variable, and $v_1 = (1, -2)$ is a nonzero solution. Thus $v_1$ represents an eigenvector belonging to and spanning the eigenspace of $\lambda = 4$

Subtract $\lambda = 1$ down the diagonal of $A$, we have

$M = A - I = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}$, corresponding to $\begin{cases} x - y = 0 \\ -2x + 2y = 0 \end{cases}$

The system has only one free variable, and $v_2 = (1, 1)$ is a nonzero solution. $v_2$ represents an eigenvector belonging to and spanning the eigenspace of $\lambda = 1$

## (b)

Let $P$ be the matrix whose columns are $v_1$ and $v_2$.

$P = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$ and $P^{-1} = \begin{bmatrix} 1/3 & -1/3 \\ 2/3 & 1/3 \end{bmatrix}$

The diagonal matrix $D = P^{-1}AP = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$

The diagonal entries of $D$ are the eigenvalues of $A$ corresponding to the eigenvectors appearing in $P$

## (c)

Using the diagonal factorization $A = PDP^{-1}$, and $4^6 = 4096$ and $1^6 = 1$,

$$A^6 = PD^6P^{-1} = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 4096 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/3 & -1/3 \\ 2/3 & 1/3 \end{bmatrix} = \begin{bmatrix} 1366 & -1365 \\ -2730 & 2731 \end{bmatrix}$$

$f(4) = 45$ and $f(1) = 6$. Hence

$$f(A) = Pf(D)P^{-1} = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 45 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} 1/3 & -1/3 \\ 2/3 & 1/3 \end{bmatrix} = \begin{bmatrix} 19 & -13 \\ -26 & 32 \end{bmatrix}$$

## (d)

Here $\begin{bmatrix} \pm 2 & 0 \\ 0 & \pm 1 \end{bmatrix}$ are square roots of $D$. Therefore the positive square root of $A$ is

$$B = P\sqrt{D}P^{-1} = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/3 & -1/3 \\ 2/3 & 1/3 \end{bmatrix} = \begin{bmatrix} 4/3 & -1/3 \\ -2/3 & 5/3 \end{bmatrix}$$

# Question 9.48

## (a)

$$\begin{aligned}
\det(A - \lambda I) &= \begin{vmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & -3 & 3 \\ 2+\lambda & -2-\lambda & 0 \\ 6 & -6 & 4-\lambda \end{vmatrix} \\
&= (2+\lambda)\begin{vmatrix} 1-\lambda & -3 & 3 \\ 1 & -1 & 0 \\ 6 & -6 & 4-\lambda \end{vmatrix} = (2+\lambda)\begin{vmatrix} 1-\lambda & -3 & 3 \\ 1 & -1 & 0 \\ 0 & 0 & 4-\lambda \end{vmatrix} \\
&= (2+\lambda)(4-\lambda)\begin{vmatrix} 1-\lambda & -3 \\ 1 & -1 \end{vmatrix} = (2+\lambda)^2(4-\lambda)
\end{aligned}$$

Accordingly, $\lambda = 4$ and $\lambda = -2$ are eigenvalues of $A$.

Subtract $\lambda = 4$ down the diagonal of $A$, we have

$M = A - 4I = \begin{bmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{bmatrix}$, corresponding to

$$\begin{cases} -3x - 3y + 3z = 0 \\ 3x - 9y + 3z = 0 \\ 6x - 6y = 0 \end{cases} \text{ or } \begin{cases} -2x + z = 0 \\ x - y = 0 \end{cases}$$

Only one free variable, and $v_1 = (1, 1, 2)$ is a nonzero solution

Subtract $\lambda = -2$ down the diagonal of $A$, we have

$M = A + 2I = \begin{bmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{bmatrix}$, corresponding to

$$\begin{cases} 3x - 3y + 3z = 0 \\ 3x - 3y + 3z = 0 \\ 6x - 6y + 6z = 0 \end{cases} \text{ or } x - y + z = 0$$

Two free variables, and $v_2 = (1, 1, 0)$ and $v_3 = (1, 0, -1)$ are linearly independent solutions

There are altogether 3 independent eigenvectors which can generate the $P$, so $A$ can be diagonalized

## (b)

$$\begin{aligned} \det(B - \lambda I) &= \begin{vmatrix} 3-\lambda & -1 & 1 \\ 7 & -5-\lambda & 1 \\ 6 & -6 & 2-\lambda \end{vmatrix} = \begin{vmatrix} 3-\lambda & -1 & 1 \\ 4+\lambda & -4-\lambda & 0 \\ 6 & -6 & 2-\lambda \end{vmatrix} \\ &= (4+\lambda) \begin{vmatrix} 3-\lambda & -1 & 1 \\ 1 & -1 & 0 \\ 6 & -6 & 2-\lambda \end{vmatrix} = (4+\lambda) \begin{vmatrix} 3-\lambda & -1 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} \\ &= (4+\lambda)(2-\lambda) \begin{vmatrix} 3-\lambda & -1 \\ 1 & -1 \end{vmatrix} = -(2-\lambda)^2(4+\lambda) \end{aligned}$$

Accordingly, $\lambda = -4$ and $\lambda = 2$ are eigenvalues of $B$.

Subtract $\lambda = -4$ down the diagonal of $B$, we have

$M = B + 4I = \begin{bmatrix} 7 & -1 & 1 \\ 7 & -1 & 1 \\ 6 & -6 & 6 \end{bmatrix}$, corresponding to

$$\begin{cases} 7x - y + z = 0 \\ 7x - y + z = 0 \\ 6x - 6y + 6z = 0 \end{cases} \text{ or } \begin{cases} x = 0 \\ x - y + z = 0 \end{cases}$$

One free variable, and $v_1 = (0, 1, 1)$ is a nonzero solution

Subtract $\lambda = 2$ down the diagonal of $B$, we have

$$M = B - 2I = \begin{bmatrix} 1 & -1 & 1 \\ 7 & -7 & 1 \\ 6 & -6 & 0 \end{bmatrix}, \text{ corresponding to}$$

$$\begin{cases} x - y + z = 0 \\ 7x - 7y + z = 0 \\ 6x - 6y = 0 \end{cases} \quad \begin{cases} x - y = 0 \\ z = 0 \end{cases}$$

One free variable, and $v_2 = (1, 1, 0)$

There are altogether 2 independent eigenvectors, so $B$ can not be diagonalized

## (c)

$$\begin{aligned} \det(C - \lambda I) &= \begin{vmatrix} 1-\lambda & 2 & 2 \\ 1 & 2-\lambda & -1 \\ -1 & 1 & 4-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 2 & 2 \\ 0 & 3-\lambda & 3-\lambda \\ -1 & 1 & 4-\lambda \end{vmatrix} \\ &= (3-\lambda)\begin{vmatrix} 1-\lambda & 2 & 2 \\ 0 & 1 & 1 \\ -1 & 1 & 4-\lambda \end{vmatrix} = (3-\lambda)\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1 & 1 \\ -1 & 1 & 4-\lambda \end{vmatrix} \\ &= (3-\lambda)(1-\lambda)\begin{vmatrix} 1 & 1 \\ 1 & 4-\lambda \end{vmatrix} = (3-\lambda)^2(1-\lambda) \end{aligned}$$

Accordingly, $\lambda = 1$ and $\lambda = 3$ are eigenvalues of $C$.

Subtract $\lambda = 1$ down the diagonal of $C$, we have

$$M = C - I = \begin{bmatrix} 0 & 2 & 2 \\ 1 & 1 & -1 \\ -1 & 1 & 3 \end{bmatrix}, \text{ corresponding to}$$

$$\begin{cases} 2y + 2z = 0 \\ x + y - z = 0 \\ -x + y + 3z = 0 \end{cases} \text{ or } \begin{cases} y + z = 0 \\ x + 2y = 0 \end{cases}$$

One free variable, and $v_1 = (2, -1, 1)$ is a nonzero solution

Subtract $\lambda = 3$ down the diagonal of $C$, we have

$M = C - 3I = \begin{bmatrix} -2 & 2 & 2 \\ 1 & -1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$, corresponding to

$x - y - z = 0$

Two free variables, and $v_2 = (1, 1, 0)$ and $v_3 = (1, 0, 1)$ are linearly independent solutions

There are altogether 3 independent eigenvectors which can generate the $P$, so $C$ can be diagonalized.

# Question 9.49

## (a)

$T = \begin{bmatrix} 3 & 3 \\ 1 & 5 \end{bmatrix}$

$\det(T - \lambda I) = \begin{vmatrix} 3-\lambda & 3 \\ 1 & 5-\lambda \end{vmatrix} = \lambda^2 - 8\lambda + 12 = (\lambda - 2)(\lambda - 6)$

The roots of the equation give the eigenvalues, therefore the eigenvalues are $\lambda = 2$ and $\lambda = 6$.

Subtract $\lambda = 2$ down the diagonal of $T$, we have

$M = T - 2I = \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix}$, corresponding to $\begin{cases} x + 3y = 0 \\ x + 3y = 0 \end{cases}$

The system has only one free variable, and $v_1 = (3, -1)$ is a nonzero solution, an eigenvector corresponding to eigenvalue $\lambda = 2$.

Subtract $\lambda = 6$ down the diagonal of $T$, we have

$M = T - 6I = \begin{bmatrix} -3 & 3 \\ 1 & -1 \end{bmatrix}$, corresponding to $\begin{cases} -3x + 3y = 0 \\ x - y = 0 \end{cases}$

The system has only one free variable, and $v_2 = (1, 1)$ is a nonzero solution, an eigenvector corresponding to eigenvalue $\lambda = 6$.

The two vectors $v_1$ and $v_2$ form the basis for the eigenspace.

### (b)

$$T = \begin{bmatrix} 3 & -13 \\ 1 & -3 \end{bmatrix}$$

$$\det(T - \lambda I) = \begin{vmatrix} 3-\lambda & -13 \\ 1 & -3-\lambda \end{vmatrix} = \lambda^2 + 4 = (\lambda - 2i)(\lambda + 2i) = 0$$

Since the linear operators are defined on **real** space, complex eigenvalues are not allowed. Therefore there are no eigenvalues.

## Question 9.51

$$\det(A^T - \lambda I) = \det(A^T - (\lambda I)^T) = \det((A - \lambda I)^T) = \det(A - \lambda I)$$

So the eigenvalues, which are the roots of this equation are the same for $A$ and $A^T$.

An example for which $A$ and $A^T$ have different eigenvectors is:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}.$$

The two eigenvalues are 1 and 2. The eigenvectors for $A$ are $(1, 0)$ and $(1, 1)$. The two eigenvectors of $A^T$ are $(1, -1)$ and $(0, 1)$.

# Question 9.53

## (a)

$$\begin{aligned} T^n v &= T^{n-1}Tv = T^{n-1}\lambda v = T^{n-1}v\lambda \\ &= T^{n-2}Tv\lambda = T^{n-2}\lambda v\lambda = T^{n-2}v\lambda^2 \\ &= T^{n-3}Tv\lambda^2 = T^{n-3}\lambda v\lambda^2 = T^{n-3}v\lambda^3 \\ &= \cdots \\ &= Tv\lambda^{n-1} = \lambda v\lambda^{n-1} = \lambda^n v \end{aligned}$$

## (b)

Represent the polynomial $f(t)$ by $a_n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$, therefore

$$\begin{aligned} f(T)v &= (a_nT^n + a_{n-1}T^{n-1} + \cdots + a_1T + a_0)v \\ &= a_nT^nv + a_{n-1}T^{n-1}v + \cdots + a_1Tv + a_0v \\ &= a_n\lambda^nv + a_{n-1}\lambda^{n-1}v + \cdots + a_1\lambda v + a_0v \\ &= (a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0)v \\ &= f(\lambda)v \end{aligned}$$

# Question 9.54

$(G \circ F)(G(v)) = G(F(G(v))) = G((F \circ G)(v)) = G(\lambda v) = \lambda G(v)$.

Therefore, $\lambda$ is an eigenvalue of the composition $G \circ F$ with corresponding eigenvector $G(v)$, if $\lambda$ is an eigenvalue of the composition $F \circ G$ with eigenvector $v$.