Linear Algebra, Spring 2005

Solutions

May 4, 2005

Solution to 4.71

(a) $E_1 = 4(5u - 6v) + 2(3u + v)$
$= 20u - 24v + 6u + 2v = 20u + 6u - 24v + 2v = 26u - 22v$

(b) $E_2 = 5(2u - 3v) + 4(7v + 8u)$ (Note: There is a typo in the text book)
$= 10u - 15v + 28v + 32u = 10u + 32u - 15v + 28v = 42u + 13v$

(c) $E_3 = 6(3u + 2v) + 5u - 7v$
$= 18u + 12v + 5u - 7v = 18u + 5u + 12v - 7v = 23u + 5v$

(d) $E_4 = 3(5u + 2v)$ (Note: There is a typo in the text book)
$= 15u + 6v$

Solution to 4.72

[A1]:
(i) $[(a, b) + (c, d)] + (e, f) = (a + c, b + d) + (e, f) = (a + c + e, b + d + f)$
(ii) $(a, b) + [(c, d) + (e, f)] = (a, b) + (c + e, d + f) = (a + c + e, b + d + f)$
comparing (i) & (ii) proves $[A1]$.

[A2]:
$(a, b) + (0, 0) = (a + 0, b + 0) = (a, b)$
$(0, 0) + (a, b) = (0 + a, 0 + b) = (a, b)$

[A3]:
$(a, b) + (-a, -b) = (a + (-a), b + (-b)) = (0, 0)$
$(-a, -b) + (a, b) = ((-a) + a, (-b) + b) = (0, 0)$

[A4]:
(a,b)+(c,d) = (a+c,b+d) = (c+a,d+b) = (c,d) + (a,b)

[M1]
(i): $k[(a, b) + (c, d)] = k(a + c, b + d) = (k(a + c), 0) = (ka + kc, 0)$
(ii): $k(a, b) + k(c, d) = (ka, 0) + (kc, 0) = (ka + kc, 0 + 0) = (ka + kc, 0)$
comparing (i) and (ii) proves [M1].

[M2]
$(k_1 + k_2)(a, b) = ((k_1 + k_2)a, (k_1 + k_2)b) = (k_1a + k_2a, k_1b + k_2b) = (k_1a, k_1b) + (k_2a, k_2b) = k_1(a, b) + k_2(a, b)$

[M3]
(i) $(k_1k_2)(a, b) = ((k_1k_2)a, 0) = (k_1k_2a, 0)$
(ii) $k_1(k_2(a, b)) = k_1(k_2a, 0) = (k_1k_2a, 0)$
Comparing (i) and (ii) proves [M3]

[M4]
$1(a, b) = (1a, 0) = (a, 0) \neq (a, b)$
[M4] is not satisfied.

Solution to 4.73

Proposition 1: $0u = 0$

$0u = (0 + 0)u = 0u + 0u$
$0 = 0u + (-0u) = (0u + 0u) + (-0u) = 0u + (0u + (-0u)) = 0u$

Proposition 2: $-u = (-1)u$

$$\begin{aligned} -u &= (-u) + 0 = (-u) + 0u = (-u) + (1 + (-1))u = (-u) + (1u + (-1)u) \\ &= ((-u) + u) + (-1)u = (-1)u \end{aligned}$$

Proof for 4.73

$$\begin{aligned} u+v &= (u+v)+0=(u+v)+((-u)+u)=(u+v)+((-u)+0+u) \\ &= (u+v)+((-u)+((-v)+v)+u)=(u+v)+(-u)+(-v)+v+u \\ &= (u+v)+(-1)u+(-1)v+v+u=1(u+v)+(-1)(u+v)+v+u \\ &= (1+(-1))(u+v)+v+u=0(u+v)+v+u=v+u \end{aligned}$$

Solution to 1.41

(a)

$3u-2v=3(1,-2,4)-2(3,5,1)=(3,-6,12)-(6,10,2)=(-3,-16,10)$

(b)

$5u+3v-4w=5(1,-2,4)+3(3,5,1)-4(2,1,-3)=(5,-10,20)+(9,15,3)-(8,4,-12)=(6,1,35)$

Solution to 1.43

(a)

$4u-3v=4(2,-5,4,6,-3)-3(5,-2,1,-7,-4)=(8,-20,16,24,-12)-(15,-6,3,-21,-12)=(-7,-14,13,45,0)$

(b)

$5u+2v=5(2,-5,4,6,-3)+2(5,-2,1,-7,-4)=(10,-25,20,30,-15)+(10,-4,2,-14,-8)=(20,-29,22,16,-23)$

Solution to 1.46

(a)

$(x,y+1)=(y-2,6)\Rightarrow$

$$\begin{cases} x &= y-2 \\ y+1 &= 6 \end{cases}$$

$\Rightarrow$

$$\begin{cases} x &= y-2 \\ y &= 5 \end{cases}$$

$$x = 5 - 2 = 3$$

(b)

$x(2, y) = y(1, -2) \Rightarrow (2x, xy) = (y, -2y) \Rightarrow$

$$\begin{cases} 2x &= y \\ xy &= -2y \end{cases}$$

$$x = -2, y = -4$$

Solution to 1.47

$(x, y+1, y+z) = (2x+y, 4, 3z) \Rightarrow$

$$\begin{cases} x &= 2x+y \\ y+1 &= 4 \\ y+z &= 3z \end{cases}$$

$$\begin{cases} -x &= y \\ y &= 3 \\ y &= 2z \end{cases}$$

$x = -3, y = 3, z = 1.5$

Solution to 1.66

(a)

$(4-7i)(9+2i) = 36 + 8i - 63i - 14i^2$ =36-14(-1)-55i = 50-55i

(b)

$(3-5i)^2 = 9 - 30i + 25i^2 = 9 - 25 - 30i = -16 - 30i$

(c)

$\frac{1}{4-7i} = \frac{4+7i}{(4-7i)(4+7i)} = \frac{4+7i}{16-49i^2}\frac{4+7i}{16+49} = \frac{1}{65}(4-7i)$

(d)

$\frac{9+2i}{3-5i} = \frac{(9+2i)(3+5i)}{(3-5i)(3+5i)} = \frac{(27+45i+6i+10i^2)}{(9-25i^2)} = 3\frac{1}{34}(17+51i) = \frac{17}{34}(1+3i) = \frac{1}{2}(1+3i)$

(e)

$(1-i)^3 = (1-i)(1-i)^2 = (1-i)(1-2i+i^2) = (1-i)(1-2i-1) = (1-i)(-2i) = -2i + 2i^2 = -2 - 2i$

Solution to 1.67

(a)

$\frac{1}{(2i)} = \frac{i}{(2i^2)} = \frac{-i}{2}$

(b)

$\dfrac{2+3i}{7-3i} = \dfrac{(2+3i)(7+3i)}{(7-3i)(7+3i)} = \dfrac{(14+6i+21i+9i^2)}{(49-9i^2)} = \dfrac{14+27i-9}{58} = \dfrac{5}{58} + \dfrac{27}{58}i$

(c)

$i^{15} = i^{12}i^2i = 1 \times (-1)i = -i;$
$i^{25} = i^{24}i = 1 \times i = i;$
$i^{34} = i^{32}i^2 = 1 \times (-1) = -1;$

(d)

$(\frac{1}{3-i})^2 = (\frac{3+i}{(3-i)(3+i)})^2 = (\frac{3+i}{9-i^2})^2 = (\frac{3+i}{10})^2 = \frac{9+6i+i^2}{100} = \frac{9+6i-1}{100} = \frac{2}{25} + \frac{3}{50}i$

Solution to 1.68

(a)

$z + w = (2-5i) + (7+3i) = 9 - 2i$

(b)

$zw = (2-5i)(7+3i) = 14 - 35i + 6i - 15i^2 = 29 - 29i$

(c)

$z/w = \frac{2-5i}{7+3i} = \frac{(2-5i)(7-3i)}{(7+3i)(7-3i)} = \frac{14-35i-6i+15i^2}{58} = -\frac{1}{58} - \frac{41}{58}i$

(d)

$\overline{z} = 2 + 5i$, $\overline{w} = 7 - 3i$

(e)

$|z| = \sqrt{2^2+5^2} = \sqrt{29}$, $|w| = \sqrt{7^2+3^2} = \sqrt{58}$

Solution to 1.69

(Note: In the following we assume that z can be written as: $z = a + bi$)

(a)

$\frac{1}{2}(z+\bar{z}) = \frac{1}{2}(a+bi+a-bi) = \frac{1}{2}(2a) = a = Re(z)$

(b)

$\frac{1}{2}(z-\bar{z}) = \frac{1}{2}(a+bi-a+bi) = \frac{1}{2}(2bi) = bi = Im(z)$

(c)

$zw = 0 \Rightarrow |zw| = 0 \Rightarrow |z| = 0$ or $|w| = 0 \Rightarrow z = 0$ or $w = 0$

Solution to 1.70

(a)

$u+v = (1+7i, 2-6i) + (5-2i, 3-4i) = (1+7i+5-2i, 2-6i+3-4i) = (6+5i, 5-10i)$

(b)

$(3+i)u = (3+i)((1+7i),(2-6i)) = ((3+i)(1+7i),(3+i)(2-6i)) = (-4+22i, 12-16i)$

(c)

$$\begin{aligned} 2iu+(4+7i)v &= 2i((1+7i),(2-6i)) + (4+7i)((5-2i),(3-4i)) \\ &= (2i(1+7i), 2i(2-6i)) + ((4+7i)(5-2i),(4+7i)(3-4i)) \\ &= (-14+2i, 12+4i) + (34+27i, 40+5i) \\ &= (20+29i, 52+9i) \end{aligned}$$